Differential Equations

Mathematics

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NOTICE

TABLE OF CONTENTS

I. Differential Equations

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II. Prerequisite courses or knowledge

Calculus unit 3

III. Time

The total time for this module is 120 study hours divided as shown below:

Learning activity	Topic	Unit	Time
#1	Introduction to first and second order differential equations	one	30 hours
#2	Techniques and tools for solving a variety of problems of linear differential equations	one	30 hours
#3	Series solutions of second order linear differential equations	two	30 hours
#4	Partial differential equations; Laplace transforms, Fourier series, and their applications	two	30 hours

IV. Material

Students should have access to the core readings specified later. Also, they will need a computer to gain full access to the core readings. Additionally, students should be able to install suitable computer software wxMaxima and use it to practice algebraic concepts.

V. Module Rationale

Differential equations arise in many areas of science and technology whenever a relationship involving some continuously changing quantities and their rates of change is known or formed. For instance in classical mechanics, the motion of a body is described by its position and velocity as the time varies. Newton's Laws allow one to relate the position, velocity, acceleration and various forces acting on the body and this relation can be expressed as a differential equation for the unknown position of the body as a function of time. In many cases, the differential equation may be solved, to yield the law of motion.

Differential equations are studied from several different perspectives. Some examples where differential equations have been used to solve real life problems include the diagnosis of diseases and the growth of various populations Braun, M.(1978).First order and higher order differential equations have also found numerous applications in problems of mechanics, electric circuits, geometry, biology, chemistry, economics, engineering , and rocket science. Spiegel, M.R. (1981,pp.70-162).The study of differential equations should therefore equip the mathematics and science teachers with the knowledge and skills to teach their respective subjects well, by incorporating relevant applications in their subject areas..

VI. Content

6.1 Overview

This module consisst of two units,namely Introduction to Ordinary differential equations and higher order differential equations respectively.In unit one both homogeneous and non-homogeneous ordinary differential equations are discussed and their solutions obtained with a variety of techniques.Some of these techniques include the variation of parameters, the method of undetermined coefficients and the inverse operators. In unit two series solutions of differential equations are discussed .Also discussed are partial differential equations and their solution by separation of variables.Other topics discussed are Laplace transforms, Fourier series,Fourier transforms and their applications.

6.2 Outline

Syllabus

Unit 1: Introduction to Ordinary Differential Equations

Level 2. Priority A. Calculus 3 is prerequisite.

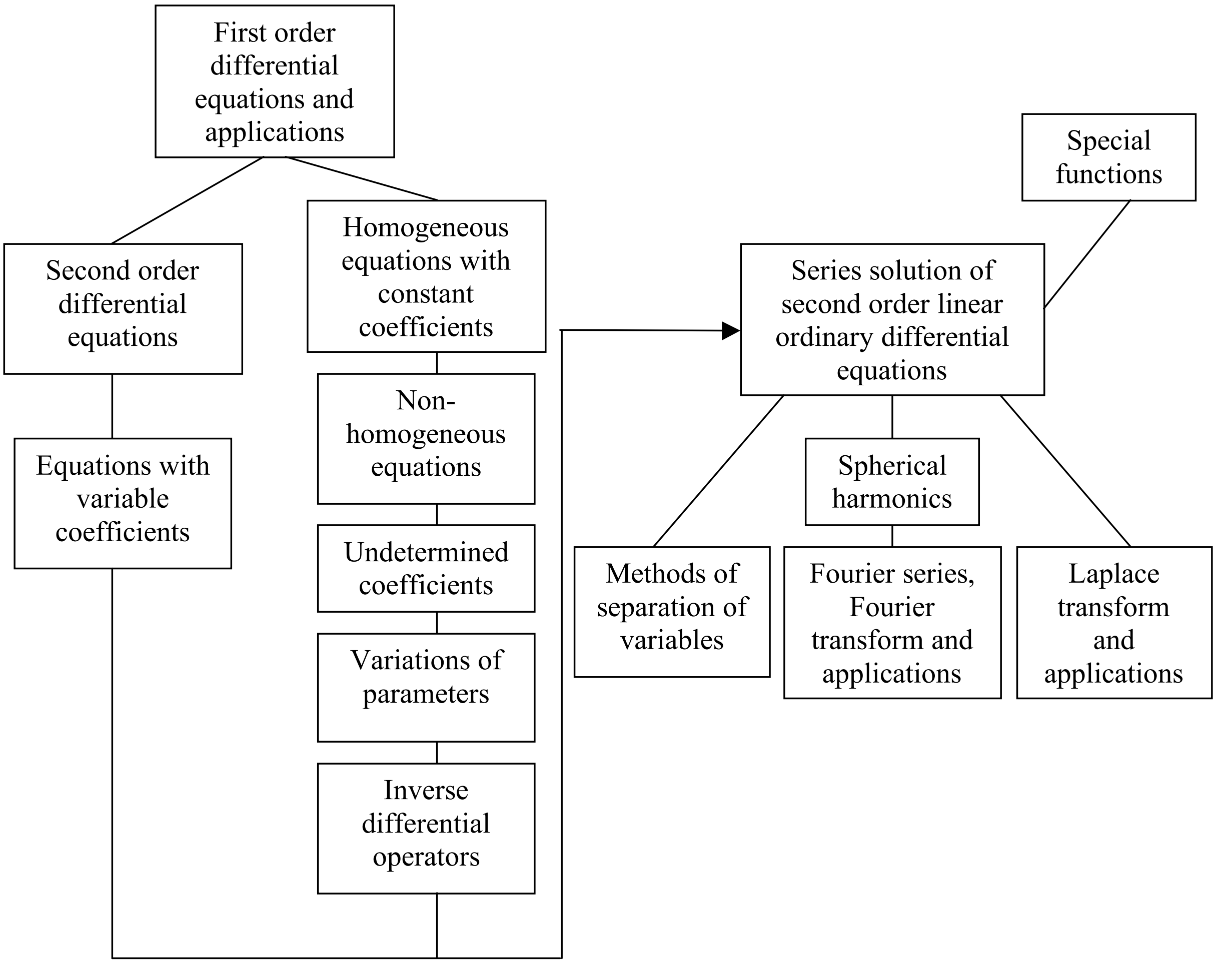
First order differential equations and applications. Second order differential equations. Homogeneous equations with constant coefficients. Equations with variable coefficients. Non-homogeneous equations. Undetermined coefficients. Variations of parameters. Inverse differential operators.

Unit 2: Higher Order Differential Equations and Applications

Level 2. Priority B. Differential Equations 1 is prerequisite.

Series solution of second order linear ordinary differential equations. Special functions. Methods of separation of variables applied to second order partial differential equations. Spherical harmonics. Laplace transform and applications. Fourier series, Fourier transform and applications

Graphic Organiser



VII. General Objective(s)
for the whole module

By the end of this module the student should be able to:

1. Demonstrate an understanding of differential equations and mastery of different techniques to apply them to solve real life problems.
2. Demonstrate an understanding of the concepts and properties of special functions, Laplace transforms, Fourier series, Fourier transforms and master their applications.
3. Exploit ICT opportunities in general and Computer Algebra Systems (CAS) in particular,to explore the algebra and solutions of differential equations.

VIII. Specific Learning Objectives
(Instructional Objectives): separate objectives for each unit

You should be able to:

1. Demonstrate an understanding of differential equations and master different techniques to apply them to solve problems.
2. Demonstrate an understanding of the concepts and properties of special functions, Laplace transforms, Fourier series, Fourier transforms and master their applications.

You should secure your knowledge of school mathematics in:

1. Basic calculus: differentiation and integration

You should exploit ICT opportunities in:

1. Using Computer Algebra Systems (CAS) to explore the algebra of differential equations.

IX. Teaching And Learning Activities

9.1 Pre-assessment

QUESTIONS

1. Which of the following trigonometric statements is **not** identically true?

A. $\sin^2 x + \cos^2 x = 1$

B. $\sec^2 x - \tan^2 x = 1$

C. $\tan(-x) = \tan x$

D. $\cos(-x) = \cos x$

2. What is the equation of the tangent line to the curve $y = x^2 - 3$ at the point (2,1) ?

A. $y = 2x - 3$

B. $y = 4x - 7$

C. $y = 4x - 9$

D. $y = 4x - 5$

3. If $y = \tan x$, then $\frac{dy}{dx}$ equals?

A. $\cot^2 x$

B. $\sec^2 x$

C. $\sec x \tan x$

D. $\operatorname{cosec} x$

4. Differentiate $f(x) = \frac{1}{2}e^{2x}$ with respect to x.

A. $\frac{1}{4}e^{2x}$ C. $\frac{1}{4}e^{x}$

B. e^{x} D. e^{2x}

5. The function $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ is the standard Taylor's series for

A. $\sin x$
B. $\cos x$
C. $\sin(-x)$
D. $\cos(-x)$

6. To find the derivative of the function $y = x \sin x$, the basic principle applied is:

A. Trigonometry
B. Quotient principle
C. Parametric principle
D. Product principle

7. Integration is sometimes described as the_______________of differentiation.

(Fill in the blank with the appropriate word)

A. process
B. reverse
C. extreme
D. result

8. To work out the solution of $\int e^{2x} \sin x dx$, the commonest approach is to apply:

A. direct integration
B. substitution method
C. partial fractions
D. integration by parts

9. Express $\frac{x^3 - 1}{x^2 - 1}$ into partial fractions

A. $x + \frac{1}{x-1}$ C. $x + \frac{1}{x+1}$

B. $x - \frac{1}{x-1}$ D. $x - \frac{1}{x+1}$

10. Find the integral of $\frac{x^3 - 1}{x^2 - 1}$

A. $\frac{x^2}{2} + \ln(x-1) + c$

B. $\frac{x^2}{2} - \ln(x-1) + c$

C. $\frac{x^2}{2} - \ln(x+1) + c$

D. $\frac{x^2}{2} + \ln(x+1) + c$

Answer key

1.	C	2.	B	3.	B	4.	D.	5.	A
6.	D	7.	B	8.	D	9.	C	10.	D

Pedagogical Comment For Learners

1. Trigonometric identities are available in most basic mathematics texts. You should cross-check these identities and take note.
2. Essentially the problem of finding the tangent line at apoint P boils down to the the problem of finding the slope of the tangent at point P.Refer to unit 1 of module 3
3. Trigonomrtric derivatives are standard expressions available in most basic mathematics texts. In some cases these derivatives are arrived at from first principles. Please refer also to unit 1 of module 3.
4. Please refer to unit 1, module 3
5. Please refer to unit 3, module 3
6. Please refer to unit 2, module 3
7. Please refer to unit 2, module 3
8. Please refer to unit 2, module 3
9. Please refer to unit 2, module 3
10. Please refer to unit 2, module 3

X. Learning Activities

Learning Activity #1

Introduction to first and second order Differential Equations

Specific learning Objectives

By the end of this unit, the learner should be able to :

- Correctly identify differential equations of various orders and degrees;
- Form a differential equation by elimination of arbitrary constants;
- Solve first order differential equation problems using the method of separation of variables; and
- Solve first order homogeneous differential equation problems by reduction to variables separable.

Summary

This unit introduces differential equations module. Prior knowledge and skills in differential and integral calculus, covered under calculus module is assumed.

In this unit , you will learn how to correctly identify differential equations by stating their order and the degrees. You will also learn how to form a differential equation from a given function. You will solve differential equation problems by method of separation of variables. Finally, you will also learn how to solve homogeneous differential equations by reduction to variables separable method.

List of Required Reading:

Mauch, S.(2004).*Introduction to Methods of Applied Differential Equations or Advanced Mathematics Methods for Scientists and Engineers*: Mauch Publishing Company.

http://www.its.caltech.edu/~sean

Additional General Reading:

Stephenson, G. (1973). Mathematical Methods for Science Students. Singapore: Longman. p.380-386.

Wikibooks, *Differential Equations*

Key Words

Differential equation: A differential equation is a relation between a function and its derivatives

Order: The order of a differential equation is an integer showing the highest ordered derivative in a given equation.

Degree: The degree of an ordinary differential equation is the power to which the highest ordered derivative is raised.

Learning Activity

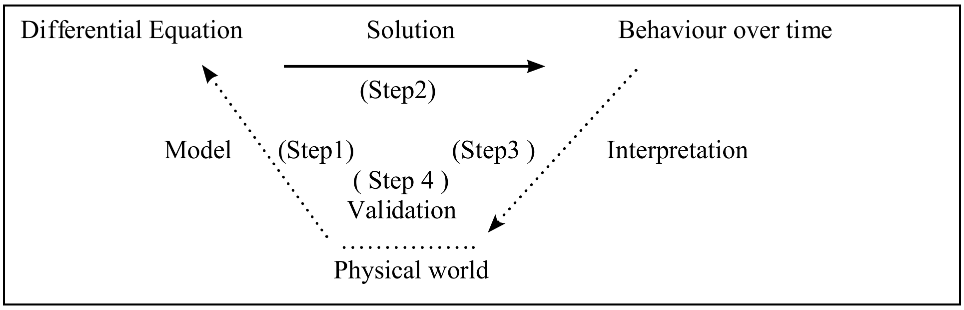
Introduction to First and Second Order Differential Equations

1.1 Differential Equations

A differential equation is a relation between a function and its derivatives. Differential equations form the language in which the basic laws of physical science are expressed.

The science tells us how a physical system changes from one instant to the next. The theory of differential equations then provides us with the tools and techniques to take this short term information and obtain the long-term overall behaviour of the system.

The art and practice of differential equations involves the following sequence of steps.



A dynamic physical system.

This pendulum illustrates how a physical system changes with time.

1.1.1 Definition: Ordinary and partial differential equations

A differential equation (DE) is an equation involving derivatives of unknown function of one or more variables. If the unknown function depends on only one variable, the equation is called an ordinary differential equation (ODE). If the unknown function depends on more than one variable, the equation is called a partial differential equation (PDE).

Example 1: $\frac{dy}{dx} = 2x + y$ or $y' = 2x + y$ is an ordinary differential equation since the function $y = f(x)$ depends on only one variable x. In the function $y = f(x)$, x is called the independent variable, and y is the dependent variable.

Example 2: $\frac{\partial y}{\partial x} = 2x + z$ is a partial differential equation since the function

$y = f(x,z)$ depends on two variables x and z.

1.1.2 Definition : Order and degree of a differential equation

The order of a DE is the order of the highest ordered derivative involved in the expression . The degree of the an ordinary differential equation is the algebraic degree of its highest ordered–derivative.

Example 3: $\frac{dy}{dx} = 2x + y$ is first order, first degree ordinary DE

Example 4: $\frac{\partial y}{\partial x} = 2x + z$ is first order, first degree partial DE

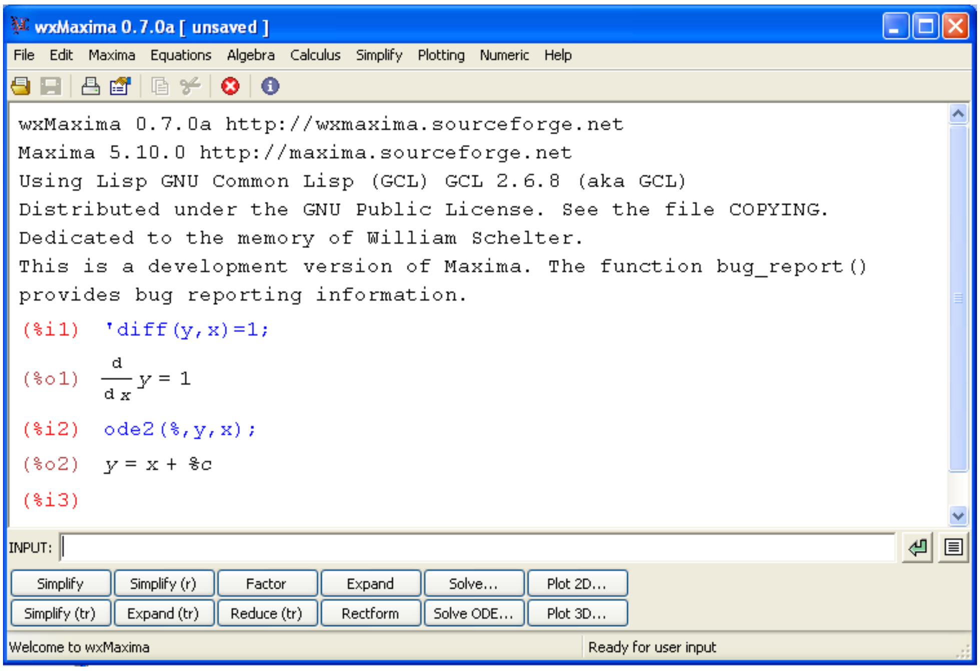
Example 5: $\frac{d^2x}{dt^2} - 2\frac{dx}{\partial t} - 15x = 0$ is second order, first degree ordinary DE

Activities 1.1.1

Software Activity: Investigate simple differential equations using wxMaxima.

wxMaxima can solve differential equations for you. You must not use this system to avoid solving exercise questions yourself! Instead, you can use it to explore different equations and think about how they work.

- First load wxMaxima.
- You type into the command line at the bottom of the screen.
- Type 'diff(y,x) and press ENTER. Notice the apostrophe ' at the start.
- This enters $\frac{d}{dx}y$ i.e. $\frac{dy}{dx}$.
- Now enter the differential equation,starting with a simpler one : $\frac{dy}{dx} = 1$. Think first: what is the solution?
- If the differential is 1, then the function must be x. With an arbitrary constant added i.e. x + C
- In wxMaxima, type: 'diff(y,x)=1 and press ENTER.
- You should see the differential equation $\frac{dy}{dx} = 1$.
- Now ask wxMaxima to solve the equation for you.
- To do this you use the **ode2** function. This stands for ordinary differential equation (up to second degree).
- You need to tell wxMaxima three things: which function you are using, what the dependent variable is, what the independent variable is.
- The function is the one you have just entered, we use % to tell wxMaxima to use this. The variables are independent: *y,* and dependent: *x*.
- So, type: ode2(%,y,x) and press ENTER.
- The solution is $y = x + \%C$
- Notice that wxMaxima shows the arbitrary constant of integration as %C.
- [You can do this all in one go by typing: ode2('diff(y,x)=1,y,x)]



Now you should experiment with different differential equations. Always decide for yourself what the answer will be **before** you press ENTER in wxMaxima!

Try these to get started:

$$\frac{dy}{dx} = 5, \qquad \frac{dy}{dx} = x, \qquad \frac{dy}{dx} = \sin x, \qquad \frac{dy}{dx} = x^2 + 5x$$

[Remember that x^2 is entered as x^2 and sin*x* must be entered as sin(x).]

Compulsory Reading: Mauch, S.(2004). pp.773-776 available on **CD .**

Using the notes in the compulsory reading and the notes given in section 1.4, discuss in a small group of 3-4 members, **order** and **degree** of the following differential equations.

In the case of the degree of differential equations given in fractions, it is advisable to rationalize the fractions first by multiplying by the lowest common denominator. Please note that the degree of the differential equation is obtained from the same term which provides the highest order in a given equation.

(i) $x^2 dy + ydx = 0$

(ii) $\left(\frac{dy}{dx}\right)^3 = 3x^2 - 1$

(iii) $\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{2}} = \left(x - \frac{dy}{dx}\right)^{\frac{1}{3}}$

(iv) $\left(\frac{d^3y}{dx^2}\right)^{2} + \left(\frac{d^2y}{dx^2}\right)^{5} + y = e^x$

1.2. Formation of a differential Equation

Although the prime problem in the study of differential equations is finding the solution to a given differential equation, the converse problem is also interesting.That is the problem of finding a differential equation which satisfies by a given solution. The problem is solved by repeated differentiation and elimination of the arbitrary constants.

Example: Find the differential equation which has $y = c_1e^x + c_2e^{-x} + 3x$ as its general solution.

Solution: Differentiate the given expression twice

(1) $y = c_1e^x + c_2e^{-x} + 3x$

(2) $y^/ = c_1e^x - c_2e^{-x} + 3$

(3) $y^{//} = c_1e^x + c_2e^{-x}$

Eliminate c_1 and c_2 by subtracting (3) from (1) to obtain $y - y^{//} = 3x$ gives which is the desired differential equation.Note that thje desired differential equation is free from the arbitrary constants.

Activity 1.2.1

Find the differential equations with the following general solutions

Hint: Use Example in section 1.2.

1. $y^2 = 4c(x + c)$, c is an arbitrary constant. [Ans: $y(y^/)^2 + 2xy^/ - y = 0$]
2. $y = Ae^{-x} + Be^{-3x}$, A and B are arbitrary constants. [Ans: $y^{//} - 4y^/ + 3y = 0$]

1.3. Solutions of First Order Differential Equations

The systemic development of techniques for solving differential equations logically begins with the equations of the first order and first degree.

Equations of this type can in general, be written as $\frac{dy}{dx} = F(x, y)$, (1.3.)

where $F(x, y)$ is a given function. However, despite the apparent simplicity of this equation, analytic solutions are usually possible only when $F(x, y)$ has simple forms. Two such forms are introduced in this activity.

1.3.1. Variables separable

Compulsory Reading: Mauch, S.(2004).pp.780-782 available on **CD**

If $F(x, y) = f(x)g(y)$ (1.3.1a)

where $f(x)$ and $g(x)$ are respectively functions of x only and y only, then, (1.3)

becomes $\frac{dy}{dx} = f(x)g(y)$ (1.3.1b).

Since the variables x and y are now separable , we have ,from (1.3.1b),

$$\int \frac{dy}{g(y)} = \int f(x)dx, \quad (1.3.1c).$$

which expresses y implicitly in terms of x.

Example: Solve the equation $\frac{dy}{dx} = \frac{y+1}{x-1}$, (1.3.1d).

Solution: Rewriting (1.3.1d) in the form of (1.3.1c),

$$\int \frac{dy}{y+1} = \int \frac{dx}{x-1}, \quad (1.3.1e).$$

or $\log_e(y+1) = \log_e(x-1) + \log_e C$ (1.3.1f)

where C is an arbitrary constant. Hence $\frac{y+1}{x-1} = C$, (1.3.1g)

is the general solution.

Activity 1.3.1

Given the boundary condition that $y = 1$ at $x = 0$.

Use the expression for the general solution in (1.3.1g) to *work out* the particular solution.

[Solution: $y = 2(1 - x) - 1)$].

1.3.2. Homogeneous differential equation

Compulsory Reading: Mauch, S.(2004).pp.786-791 available on **CD**

An expression of the *n*th degree in x and y is said to be homogeneous of degree *n*,if when x and y are replaced by tx and ty ,the result will be the original expression multiplied by t^n ; symbolically $f(tx, ty) = t^n f(x, y)$.

Example: Show that $x^2 + xy - y^2$ is homogeneous and determine the degree.

Solution: Replace x and y by tx and ty respectively, to obtain

$t^2x^2 + (tx)(ty) - t^2y^2 = t^2(x^2 + xy - y^2)$. The degree is 2.

Consider the differential equation $M(x, y)dx + N(x, y)dy = 0$.

The equation is said to be homogeneous in x and y if *M* and *N* are homogeneous functions of the same degree in x and y. The technique for solving this equation is to make the substitution $y = vx$ or $x = vy$ and is based upon the following theorem.

Theorem: *Any homogeneous differential of the first order and first degree can be reduced to the type of variables separable by a substitution of* $y = vx$ *or* $x = vy$.

Example: Find the general solution of the differential equation $2xydx - (x^2 - y^2)dy = 0$

Solution: A simple check reveals that the equation is homogeneous(Refer to the first example in this section).

Let $y = vx$ and $dy = vdx + xdv$, and substitute into the differential equation to obtain

$$2x(vx)dx - (x^2 - v^2x^2)(vdx + xdv) = 0$$

$$x^2(v + v^3)dx + x^3(v^2 - 1)dv = 0.$$

Dividing by $x^3(v + v^3)$ separates the variables:

$$\frac{dx}{x} + \frac{(v^2 - 1)dv}{v(1 + v^2)} = 0$$

Integration yields

$$\ln x - \ln v + \ln(v^2 + 1) = \ln c \text{ or } x(v^2 + 1) = cv$$

Rewriting the original variables by substituting $v = \frac{y}{x}$, we obtain $y^2 + x^2 = cy$ as the general solution.

Activity 1.3.2

Group discussion:

In this activity you will work in a group of 4-5 members. Each member of the group will read Mauch, S. (2004).pp.786-791 available on CD. Using the information from the compulsory reading , you will first try the given problem individually. When all members of the group are ready, you will meet and discuss your answers. Each member will take five minutes to present his or her solution while other members take note. Members are free to ask each other questions where they need clarification.

Problem: $2xy\frac{dy}{dx} = x^2 + y^2$, given that $y(1) = 0$.

Hint: Write y = vx implying that dy =vdx + vdx. Convert the resulting equations to first order separable equation in terms of v and x and solve. Then substitute v=y/x to obtain the required solution in terms of x and y.

[solution: $x^2 - y^2 = x$].

Additional Activities for the Unit: Group Work

These learning activities may be taken only when you are sure you have extra time on your hands to take them. You may also take them provided you answered the previous questions correctly.

Caution to the student:

You are advised to avoid the temptation of looking at the solutions given at the end before actually writing your answers down on paper.

Fill in the blanks for each of the Differential Equations.. After completing the exercise.

	Differential equation	Ordinary or Partial	Order	Degree	Independent variable	Dependent variable
1	$y' = x^2 + 5y$					
2	$xy'' - 4y' - 5y = e^{3x}$					
3	$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{dx^2} + \frac{\partial u}{dy}$					
4	$\left(\frac{d^3s}{dt^3}\right)^2 + \left(\frac{d^2s}{dt^2}\right) = s - 3t$					

Solutions to Additional Learning Activities

	Differential equation	Ordinary or partial	Order	Degree	Independent variable	Dependant variable
1	$y' = x^2 + 5y$	Ordinary	1	1	x	y
2	$y'' - 4y' - 5y = e^{3x}$	Ordinary	2	1	x	y
3	$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{dx^2} + \frac{\partial u}{dy}$	Partial	2	1	x, y, t	u
4	$\left(\frac{d^3s}{dt^3}\right)^2 + \left(\frac{d^2s}{dt^2}\right) = s - 3t$	Ordinary	3	2	t	s

Pedagogical Comments on the Solutions to additional learning activities.

Q1. This is first order ordinary differential equation.. The $y^{/}$= dy/dx tells you that the order is 1. The degree is also one because $(y^{/})^1=y^{/}$.The independent variable is x.

Q2. This is second order ordinary differential equation.. The $y^{//}= d^2y/dx^2$ tells you that the order is 2. The degree is also one because $(y^{//})^1=y^{//}$.The independent variable is again x .

Q3. This is second order partial differential equation.. The $u^{//} = d^2u/dt^2$ tells you that the order is 2. The degree is also one because $(u^{//})^1=u^{//}$.The independent variables this time are x, y, and t.

Q4. This is third order ordinary differential equation.. The $s^{///} = d^3s/dt^3$ tells you that the order is 3. The degree is given by the power to which the highest derivative $(s^{///})$ is raised. The degree is therefore two because of $(s^{///})^2$. The independent variable is t.

References

Zwillinger, D(1997).*Handbook of Differential Equations*..(3rd Ed).Boston: Academic Press.

Polynin, A.D.,& Zaitsev, V.F.,(2003).*Handbook of Exact Solutions for Ordinary Differential Equations* (2nd Ed).Boca Raton: Chapman & Hall/CRC Press. ISBN 1-58488-297-2.

Johnson, W.(1913)*A Treatise on Ordinary and partial Differential Equations.* University of Michigan Historical Math Collection: John Wiley and Sons.

Wikibooks, *Differential Equations*

Ince, E.L.(1956). *Ordinary Differential Equations.* Dover Publications,

External Links

- *Lectures on differential equations* MIT Open CourseWare video
- *Online Notes/Differential Equations* Paul Dawkins, Lamar University
- *Differential Equations, S.O.S Mathematics*
- *Introduction to modeling via differential equations* Introduction to modeling by means of differential equations, with critical remarks.
- *Differential Equation Solver* Java applet tool used to solve differential equations

Learning Activity #2

Techniques and tools for solving a variety of problems of linear differential equations

Specific learning objectives

By the end of this unit, you should be able to:

- Identify and solve problems of differential equations with variable coefficients;
- Identify and solve problems of non-homogeneous differential equations;
- Apply the method of undetermined coefficients to differential equations;
- Apply the method of variation of parameters to problems of differential equations; and
- Apply the inverse differential operator to the solution of linear differential equations.

Summary

In this unit differential equations with variable coefficients are introduced. Non-homogeneous equations are discussed. Equations with undetermined coefficients are discussed, and also the method of solution by variation of parameters. Finally, the inverse technique to the solution of differential equations is discussed. The learning activities in this unit include self study, reading assignments, group discussions, and problem solving.

Compulsory Reading (Core Text):

In this learning activity your major reference text is Mauch, S.(2004,Chapter 17).

Additional General Reading:

Wikibooks, *Differential Equations (include the specific webpage/ site)*

Key Words

Variable coefficients: Unlike differential equations with constant coefficients, there are also some differential equations with variable coefficients. Essentially, a variable coeffiecient is one that is not constant i.e. it is expressed in functional form.

Non-homogeneous equations: An homogeneous differential equation is one with the right hand side equated to zero. A non- homogeneous differential equation is an equation with the right hand side not equal to zero.

Undetermined coefficients: These are constants to be explicitly determined by solving the particular integral of a differential equation. The method that does this is called the method of undetermined coefficients.

Variation of parameters: This is a method of finding a particular solution of a linear differential equation when the general solution of the reduced equation(homogeneous equation) is known. (See notes below for details)

Inverse technique: This technique is applied in solving differential equations using the properties of a differential operator.(See notes).

2. Learning Activity

Techniques and tools for solving a variety of problems of linear differential Equations

2.1.1 Definition: Linear Equations

The linear first order equation discussed in Unit 1, above is a special case of the general linear equation of order *n*

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x) \quad (2.1.1)$$

where $a_0(x),\ a_1(x)\ \ldots\ a_n(x)$ and $f(x)$ are given functions of x or are constants.

2.1.2 Definition: Homogeneous and non-homogeneous equations

Consider equation (2.1.1).If $f(x) = 0$, it is called **homogeneous** equation with variable or constant coefficients, depending on whether $a_0(x),\ a_1(x)\ \ldots\ a_n(x)$ are functions of

of x or are constants. It is also called the **reduced** linear differential equation.

Example: $x\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 3y = 0$,is a second order homogeneous equation with variable coefficient.

If $f(x) \neq 0$ in equation (2.1.1), it is called non- homogeneous equation with variable or constant coefficients, depending on whether $a_0(x),\ a_1(x)\ \ldots\ a_n(x)$ are functions of x or are constants.Example: $x\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 3y = \sin x$ is a second order non-homogeneous equation with variable coefficient.

Activity 2.1.2

Group work. Work with a colleague on these problems. Discuss your solutions with them.Using equation (2.1.1) construct linear differential equations with the following coefficients:

(i) $a_0 = 2x^2$, $a_1 = x$. $a_2 = 2$, $a_3 = 5$, $f(x) = \cos x$

(ii) $a_0 = 2x^2$, $a_1 = x$. $a_2 = 2$, $f(x) = 3x^3$

2.1.3 Definition: Solution of second order differential equation

Suppose y_1 and y_2 are two independent solutions of the reduced equation of (2.1.1), namely

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad (2.1.3a)$$

then the linear combination $y = c_1 y_1 + c_2 y_2$ where c_1, c_2 are arbitrary constants, is also a solution.

Proof:

Substitute $y = c_1 y_1 + c_2 y_2$ into the equation (2.1.3a), we have

$$[a_0(x)\frac{d^n y_1}{dx^n} + a_1(x)\frac{d^{n-1} y_1}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy_1}{dx} + a_n(x)y_1] +$$
$$[a_0(x)\frac{d^n y_2}{dx^n} + a_1(x)\frac{d^{n-1} y_2}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy_2}{dx} + a_n(x)y_2] \equiv 0 \qquad (2.1.3b)$$

Equation (2.1.3b) vanishes identically to zero, since each bracket is zero by virtue of y_1 and y_2 being solutions of (2.1.3a).

2.1.4 Generalization of definition 2.1.3 to the solution of linear differential equation

Theorem 2.1.4a: If $y_1, y_2, ..., y_n$ are *n* linearly independent functions of x which satisfy a homogeneous equation (2.1.3), then the linear combination

$$y_c = c_1 y_1 + c_2 y_2 + ... + c_n y_n \tag{2.1.4}$$

where $c_1, c_2, ..., c_n$ are arbitrary constants, is its solution. Equation (2.1.4), which provides the solution for the homogeneous equation is called the complementary function.

Theorem 2.1.4b: The general solution of a complete non-homogeneous differential equation is equal to the sum of its *complementary function* and any *particular integral.* If P is a particular solution of (2.1.1), then the general solution is

$$y = y_c + P = c_1 y_1 + c_2 y_2 + ... + c_n y_n + P \tag{2.1.4b}$$

Hence, for non-homogeneous equations:

General Solution = Complementary Function + Particular

2.1.5 Application of theorem 2.1.4b

The general linear equation (2.1.4b) discussed in section 2.1.4 above is usually difficult to solve and requires special techniques. However an important and special case occurs when the coefficients $a_0(x),\ a_1(x)\ ...\ a_n(x)$ are constants, the equation being called a constant coefficient equation. Consider a constant coefficient homogeneous equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + ... + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{2.1.5a}$$

Denoting, $D_x^n y = \dfrac{d^n y}{dx^n}$, in equation (2.1.5a) we have

$$a_0 D_x^n y + a_1 D_x^{n-1} y + ... + a_{n-1} D_x y + a_n y = 0$$

$$\Rightarrow (a_0 D_x^n + a_1 D_x^{n-1} + ... + a_{n-1} D_x + a_n) y = 0 . \tag{2.1.5b}$$

If we make a formal substitution $D_x = m$ in (2.1.5b), we obtain a polynomial in m of degree n given

$$g(m) = a_0 m^n + a_1 m^{n-1} + \ldots + a_{n-1} m + a_n = 0, \qquad (2.1.5c)$$

and if we equate equation (2.1.5c) to zero we have an algebraic equation of degree n which must have n roots. The equation $g(m) = 0$ is called the **auxiliary equation** of the differential equation of the differential equation (2.1.5c).

Theorem 2.1.5: If m_1 is a root of the auxiliary equation

$a_0 m_1^n + a_1 m_1^{n-1} + \ldots + a_{n-1} m_1 + a_n = 0$, then $y = e^{mx}$ is a solution of the homogeneous linear differential equation

$(a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n) y = 0$ where a_i is a constant.

Proof: By successive differentiation we obtain

$$y = e^{m_1 x}$$
$$y' = m_1 e^{m_1 x}$$
$$y'' = m_1^2 e^{m_1 x}$$
$$y''' = m_1^3 e^{m_1 x}$$
$$\ldots$$
$$y^{(n)} = m_1^n e^{m_1 x}$$

Substitutiung these derivatives into the differential equation, we obtain

$$a_0 m_1^n e^{m_1 x} + a_1 m_1^{n-1} e^{m_1 x} + \ldots + a_{n-1} m_1 e^{m_1 x} + a_n e^{m_1 x} = 0$$

or $(a_0 m_1^n + a_1 m_1^{n-1} + \ldots + a_{n-1} m_1 + a_n) e^{m_1 x} = 0$

and since m_1 is a root of the auxiliary equation, the expression in the parenthesis is equal to zero, and the equation is satisfied.

2.1.6 Summary to solving a homogeneous differential equation

The solution of the differential equation then reduces to solving the algebraic auxiliary equation for its n roots and forming a linearly independent combination

$c_i\ e^{m_i x}\ (i = 1,\ldots,n)$ as the general solution, *if the roots were all distinct.*

Example: Find the general solution of $y'' - y' - 6y = 0$

Solution: The auxiliary equation is $m^2 - m - 6 = 0$

$(m - 3)(m + 2) = 0$, which has the roots $m = 3$ or $m = -2$. The solution is

$y = c_1\ e^{3x} + c_2\ e^{-2x}$

The student should check the answer.How can this be done? Hint: Do you recall the exercises you did on forming a differential equation in learning activity one?

Theorem 2.1.6

If the auxiliary equation of a homogeneous linear differential equation contains r as an

s-fold root, then $y = (c_0 + c_1 x + c_{2.} x^2 + \ldots + c_{s-1} x^{s-1})e^{rx}$ is a solution of the differential equation

Example: If the $r = m$ (twice), the solution is $y = (c_0 + c_1 x)e^{mx}$

2.1.7 Auxiliary equation with complex roots

If the auxiliary equation with real coefficients contains two complex roots $m_1 = a + bi$ and $m_2 = a - bi$ then $y = e^{ax}(C_1 \cos bx + C_2 \sin bx)$ is a solution of the differential equation, C_1, C_2, a, b are constants.

Generalization

If the complex roots occur as multiple roots, that is if ($(a \pm bi)$ is an s-fold pair of complex roots, then the corresponding terms of the complementary function are $y = e^{ax}[(C_0 + C_1 x + C_2 x^2 + ... + C_{s-1} x^{s-1}) \cos bx +$ $(D_0 + D_1 x + D_2 x^2 + ... + D_{s-1} x^{s-1}) \sin bx]$

Learning Activities 2.1.7

(i) Individual Reading: Read through Mauch, S.(2004).Chapter 17

(ii) Problem Solving :

Write the auxiliary equations for the following differential equations and hence solve the equations:

(a) $y'' + 3y' + 2y = 0$ *[Solution:* $y = Ae^{-x} + Be^{-2x}$ *]*

(b) $\dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$ *[Solution:* $y = (A + Bx)e^{3x}$ *]*.

Hint: Refer to Theorem 2.2.1 and the example following for some hints to this problem.

(c) $y''' - 3y'' + 7y' - 5y = 0$

Hint: Refer to the generalization in section 2.1.7.

(iii) Group Discussion

Discuss your solutions to questions **(ii)** above in small groups and see whether or not they match with the solution suggested in the brackets.

2.1.8 Equations with undetermined coefficients

In the previous section we learnt that solution to the complete linear differential equation is composed of the sum of the complementary function and the particular integral. Techniques for obtaining the complementary function y_c were developed in sections 2.1.4-2.1.6 with many working examples. What remains is only to provide techniques for finding a particular integral in order to obtain a complete solution. In this section we discuss the technique called the method of undetermined coefficients.

Although the method of undetermined coefficients is not applicable in all cases, it may be used if the right- hand side $f(x)$,contains only terms which have a finite number of linearly independent derivatives such as x^n, e^{mx} , $\sin bx$, $\cos bx$ or products of these.

2.2 Procedure for the techniques of undetermined coefficients

The general procedure in this technique is to assume the particular integral y_p of a form similar to the right member $f(x)$ in equation (2.1.1).The necessary derivatives of y_p are then obtained and substituted into the given differential equation. This results in an identity for the independent variable, and consequently the coefficients of the like terms are equated.The values of the undetermined coefficients are found from the resulting system of linear equations.The procedure is best illustrated with an example

Table 2.2.1 below summarizes a general rule for the formulation of the particular integral

If $f(x)$ *is of the form*	*Choose* y_p *to be*
$c_0 + c_1x + c_{2.}x^2 + ... + c_nx^n$	$C_0 + C_1x + C_{2.}x^2 + ... + C_nx^n$
$(c_0 + c_1x + c_{2.}x^2 + ... + c_nx^n)e^{rx}$	$(C_0 + C_1x + C_{2.}x^2 + ... + C_nx^n)e^{rx}$
$c_0 \sin bx + c_1 \cos bx$	$C_0 \sin bx + C_1 \cos bx$

Table 2.2.1

Example 2.2.1: Find the particular integral of

$y'' + 3y' + 2y = e^{2x}$

Solution: In the learning activity of section 2.1, you actually worked out the complementary function of this equation to be $y = Ae^{-x} + Be^{-2x}$.That is, the auxiliary equation is $m^2 + 3m + 2 = 0 \Rightarrow (m+2)(m+1) = 0$, giving $\Rightarrow m = -2$ *or* $\Rightarrow m = -1$

.Hence the complementary function is $y_c = Ae^{-x} + Be^{-2x}$ as before.

Looking at the right hand side of the above differential equation example 2.2.1 and the general rule in table 2.2.1, the particular integral is

$Y = Ae^{2x}, \; Y' = 2Ae^{2x}, Y'' = 4Ae^{2x}$

Substituting in the original equation $(4A + 6A + 2A)e^{2x} = e^{2x}$

Comparing the coefficients on both sides $12A = 1 \Rightarrow A = \dfrac{1}{12}$

The general solution is then the complementary function + particular integral, which is

$$y = Ae^{-x} + Be^{-2x} + \frac{1}{12}$$

Learning Activity 2.2.1

(i) Reading: Study the material presented in section 2.2.

(ii) Group Discussion

Use the background reading from section 2.2 and generate ideas how to work the general solutions the following differential equations. Compare your solutions with the ones provided in the learning activity. Do your solutions agree with the ones provided?

(a) $y'' - 5y' + 6y = x^2$ *[General solution:*

$y = Ae^{2x} + Be^{3x} + (1/6)x^2 + (5/18)x + (9/108)$*]*

(b) $y'' + 4y = 3\sin x$ *[General solution:*

$y = A\cos 2x + B\sin 2x - (3/4)x\cos 2x$*]*

2.3 Method of solution by variation of parameters (VOP)

In this learning activity your major reference text is Mauch, S.(2004, pp.795-796).

2.3.1 Introduction

The method of undetermined coefficients discussed in the previous section is limited in its application.We need another technique with wider application. The technique discussed in this section is called the method of variation of parameters.

2.3.2 Description of method

The VOP procedure consists of replacing the constants in the complementary function by undetermined functions of the independent variable x and then determining these functions so that when the modified complementary function is substituted into the differential equation, $f(x)$ will be obtained on the left side. This places only one restriction on the n arbitrary functions c_i , $(i = 1,...,n)$, and we have $(n-1)$ conditions at our disposal. We utilize this freedom in the following manner:

(a) As we differentiate y_c to find $D_x y_c = \frac{dy_c}{dx}$,there will now appear terms which contain $c_i^{/}(x)$. We set this combination of terms to zero.

(b) As we differentiate again to find $D_x^2 y_c$,we again set the resulting combination of terms containing $c_i^{/}(x)$ equal to zero.

(c) We continue this process through $D_x^{n-1} y_c$

(d) We then find $D_x^n y_c$ and substitute all these values into the given differential equation. Since y_c is the complementary function, the results of this substitution will contain only the terms of $D_x^n y_c$ which appear because the c_i are functions of x.

(e) The equations obtained from (d) and the $(n-1)$ conditions imposed by (a)-(c) will yield a system of n linear equations in n unknowns $c_i^{/}$, $(i = 1,...,n)$. This is solved and integrated to yield the n functions $c_i(x)$.

The procedure is not too difficult provided the order of the differential equation is small. The following example illustrates the technique.

Example 2.3.2

Find the general solution of $y^{//} - y = x^2$ (2.3.2)

Solution: Auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = 1$ or $m = -1$

Referring to the discussion in section 2.2 the complementary function of the differential equation is

$y_c = k_1 e^x + k_2 e^{-x}$ (2.3.2a)

Let $c_i, (i = 1,2)$ be functions of x :

$$y_p = c_1(x)e^x + c_2(x)e^{-x} \qquad (2.3.2b)$$

Differentiate to obtain $y^{/}{}_p = c_1 e^x - c_2 e^{-x} + c_1^{/} e^x + c_2^{/} e^{-x}$ (2.3.2c)

Impose the first condition i.e. $c_1^{/} e^x + c_2^{/} e^{-x} = 0$ (2.3.2d)

With condition (2.3.2d) imposed on (2.3.2c), differentiate again

to obtain $y_p^{//} = c_1 e^x + c_2 e^{-x} + c_1^{/} e^x - c_2^{/} e^{-x}$ (2.3.2e)

Substitute (2.3.2e) and (2.3.2b) into (2.3.2) giving:

$c_1 e^x + c_2 e^{-x} + c_1^{/} e^x - c_2^{/} e^{-x} - c_1 e^x - c_2 e^{-x} = x^2$,

or $c_1^{/} e^x - c_2^{/} e^{-x} = x^2$ (2.3.2f)

Note that all the terms except those containing the derivatives of $c_i, (i = 1,...,n)$, disappear and that much time can be saved by simply setting that part of (2.3.2e) equal to $f(x)$. Equations (2.3.2d) and (2.3.2f) now form a system of two linear equations

to be solved simultaneously for $c_1^{/}$ and $c_2^{/}$. Adding these two equations we obtain

$2c_1^{/} e^x = x^2$

or $$dc_1 = \frac{1}{2} x^2 e^{-x} dx$$

$$c_1 = \int \frac{1}{2} x^2 e^{-x} dx$$

Integration by parts yields

$$c_1 = -(1 + x + \frac{1}{2} x^2) e^{-x})$$

From equation (2.3.2d), we have

$$c_2^{/} = -c_1^{/} e^{2x} = -\frac{1}{2} x^2 e^x$$

Again integration by parts yields

$$c_2 = -(1 - x + \frac{1}{2} x^2) e^x$$

The general solution is then as usual the sum of the complementary function plus the particular intergral.i.e. $y = k_{1.} e^x + k_2 e^{-x} + c_1(x) e^x + c_2(x) e^{-x}$

$$= k_{1.} e^x + k_2 e^{-x} + -[1 + x + (1/2)x^2] + -[1 - x + (1/2)x^2]$$

$$= k_{1.} e^x + k_2 e^{-x} - x^2 - 2$$

Learning Activities 2.3

(i) Problem solving: Apply the techniques of variation of parameters (VOP) discussed in section 2.3 to the problems below.Note that these problems were also solved using another method section 2.2:

(a) $y^{//} - 5y^{/} + 6y = x^2$

(b) $y^{//} + 4y = 3\sin x$

Find out if VOP leads to the same solutions as obtained in section 2.2.

(ii) Group discussion

Which method do you find easier for you to apply in the given problems, and why?

2.4. Differential operators

Introduction:

In this section the theory of differential operators is outlined. The application of the theory to the solution of linear differential equations is then discussed. A number of examples are given. Together with these examples, there are also learning activities within the sections which you are to do before you proceed to the next sections.

2.4.1 Notation and Definition

To operate means to *produce an appropriate effect,* and an operator is that instrument or effect which *does* that . We have already used the notation

$$D_y^k y = \frac{d^k y}{dx^k}, k = 1,2,...$$

to indicate the kth derivative of the function y with respect to x. D^k denotes the derivative of the order k, with respect to the appropriate independent variable. D^k is called the differential operator. Since it must produce an effect, it must operate on a function and must behave according to the rules of differentiation. The following properties are valid.

Property 2.4.1a. If c is a constant $D^k(cy) = cD^k y$

Property 2. 4.1b. $D^k(y_1 + y_2) = D^k(y_1) + D^k(y_2)$

Property .2.4.1c. Two operators A and B are equal if and only if $Ay = By$.

Property 2.4.1d. If operators A, B, and C are any differential operators, they will satisfy the ordinary laws of algebra which are:

1. The commutative law of addition A+B=B+A;
2. The associative law of addition (A+B) +C= A+ (B+C);
3. The associative law of multiplication (AB) C= A (BC);
4. The distributive law of multiplication A (B+C) = AB+AC; and
5. The commutative law of multiplication if the operators all have constant coefficients AB=BA.

Property 2.4.1e. The exponential shift. If $P(D)$ is a polynomial in D with constant coefficients, then

(a) $e^{rx}P(D)y = P(D-r)[e^{rx}y]$;

(b) $P(D)[e^{rx}y] = e^{rx}P(D+r)y$;

(c) $e^{-rx}P(D)[e^{rx}y] = P(D+r)y$

2.5 Inverse Operators

To complete the discussion of the differential operator, we now consider the meaning of $D^{-k}y$ .In order to be consistent we let

$D^{-1}y = \frac{1}{D}y = z$ be an expression such that $Dz = y$.
In other words, the net effect by the differential operator with a negative index is called integration. This operator is called the **inverse differential operator.**

Definition 2.5.1 The inverse differential operator $(D-c)^{-k}$, $k = 1,2,...$, is defined as the

integral $(D-c)^{-k}y(x) = \int_{x_0}^{x} \frac{(x-u)^{k-1}}{(k-1)!} e^{c(x-u)} y(u)du$, where x_0 is an arbitrary but fixed

number.

Property 2.5.2: The folowing peoperties are relevant to the discussions in this section

Property 2.5.2a. $\frac{1}{P(D)}[e^{rx}] = \frac{e^{rx}}{P(r)}$, if $P(r) \neq 0$

Property 2.5.2b. $\frac{1}{P(D)}e^{rx} = \frac{x^k e^{rx}}{k!\varphi(r)}$.

Property 2.5.2c. $\frac{1}{D^2+r^2}\sin rx = -\frac{x}{2r}\cos rx$.**Property 2.5.2d.** $\frac{1}{D^2+r^2}\cos rx = \frac{x}{2r}\sin rx$

Property 2.5.2e. $\frac{1}{D^2+r^2}(c\sin bx)=\frac{c}{r^2-b^2}\sin bx\,,\,b\neq r$

Property 2.5.2f. $\frac{1}{D^2+r^2}(c\cos bx)=\frac{c}{r^2-b^2}\cos bx\,,\,b\neq r$

Property 2.5.2g. Illustrated by example: $\frac{1}{D(D+1)}[x^3]=\frac{x^4}{4}-3x^2+6$

Proof: $\frac{1}{D(D+1)}=D^{-1}(1-D^2+D^4-...)$

$$=D^{-1}-D^1+D^3-...$$

We have

$$\frac{1}{D(D+1)}[x^3]=D^{-1}[x^3]-D^1[x^3]+D^3[x^3]-...$$

$$=\frac{x^4}{4}-3x^2+6$$

Property 2.5.2h. Exponential Shift.

$$\frac{1}{P(D)}[e^{rx}y]=e^{rx}\frac{1}{P(D+r)}[y].$$

Learning Activity 2.5

In this learning activity your major reference text is Mauch, S.(2004, pp.902-915).

Identify the correct property from sections 2.4 and 2.5 above and use it to perform the operations below:

1. D^2e^{4x}
2. $(D^2+4)^{-1}\sin 2x$ *(Hint: Try property 2.5.2c)*
3. $[D(D^2-1)]^{-1}5x^4$ *(Hint: Try property 2.5.2g)*

2.6 Application of the inverse differential operator to the solution of linear differential equations

The use of the theory of operators can save some labour in finding particular integrals for the complete linear differential equation:

$$P(D)y = f(x) \qquad (2.6.1)$$

If we treat (2.6.1) as a simple algebraic equation, you only then have to solve for y by a

division $y = \frac{1}{P(D)} f(x)$ (2.6.2)

The properties in sections 2.5.1 and 2.5.2 can now be put to good use to our advantage

Example. Find a particular integral

$$D(D-2)^3(D+1)y = e^{2x}$$

Solution: Using the example of equation (2.6.2), solve for y to obtain the particular

intergral

$$y_p = [D(D-2)^3(D+1)]^{-1} e^{2x}$$

By property (2.5.2b), with $r = 2$, $k = 3$, and $\varphi(D) = D(D+1)$

so that

$\varphi(r) = r(r+1)$ = (2) (3) = 6, and we have

$$y_p = \frac{x^3 e^{2x}}{3!6} = \frac{1}{36} x^3 e^{2x}$$

Learning activity 2.6: Group Work

In this activity you will discuss the solution in a small group of 3-5 members. Part of the learning activity is first to identify the correct property from sections 2.4 and 2.5 for each question.

Please do bear in mind that the two equations can also be solved by other methods you have so far learnt. For example the method of undetermined coefficients in section 2.2.

Find the complete solution of the following differential equation:

1. $(D^2 - 4D + 4)y = xe^{2x}$

2. $(D^2 + 4)y = 4\cos 2x$

Learning Activity #3

Series Solutions of Second order linear differential Equations

Specific learning Objectives

By the end of this unit, you should be able to:

- Solve problems of differential equations with variable coefficients using power series methods.

Summary

In this unit solutions of linear differential equations by power series are discussed. Power series method is particularly applicable to solving differential equations with variable coefficients, where some of the methods discussed in the previous units may not work .

In this learning activity two methods are discussed: the method of successive differentiation and the method of undetermined coefficients. The technique of power series to solving differential equations requires some background knowledge of special functions of power series such as Taylor's series.

Reading (Core Text)

The Core text for this activity is Mauch, S (2004). *Introduction to Methods of Applied Mathematics*. This is available on the course CD.

Additional General Reading:

Wikibooks, *Differential Equations*

Key Words

Power series: A series whose terms contain ascending positive integral powers of a variable. e.g. $a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$, where the $a's$ are constants and x is a variable.

Variable coefficients: (Refer to key words in learning activity #2)

Taylor's series: In general if any function can be expressed as a power series such as $c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + ... + c_n(x-a)^n + ...$, that series is a Taylor's series.

Successive differentiation: One of the methods of finding the power series solutions of a differential equation

Undetermined coefficients: (Refer to Key words in learning activity #2)

3. 1 Learning activity
Series Solutions of Second order linear differential Equations

Until now we have been concerned with, and in fact restricted ourselves to differential equations which could be solved exactly and various applications which lead to them. There are certain differential equations which are of great importance in scientific applications but which cannot be solved exactly in terms of elementary functions by any method. For example, the differential equations:

$$y' = x^2 + y^2 \text{ and } xy'' + y' + xy = 0 \qquad (3.1)$$

cannot be solved exactly in terms of the functions usually studied in elementary calculus.

The question is, what possible way could we proceed to find the required solution, *if one existed?* One possible way in which we might begin could be to assume that the solution (if it exists) possesses a series solution. At this point it is important to introduce power series to assist us in working a solution to such problems as given in equation (3.1) above.

3.1.1 Definition Taylor's Series.

From calculus, you learnt that a function may be represented by Taylor's series

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''}{2!}(x - x_0)^2 + \ldots, \qquad (3.1.1)$$

provided all the derivatives exist at $(x - x_0)$. We further say that the function is analytic at $x = x_0$ if $f(x)$ can be expanded in a power series valid about the same point.

3.1.2 Definition of ordinary, singular, and regular points.

Consider a linear differential equation

$$[a_0(x)D_x^n + a_1(x)D_x^{n-1} + \ldots + a_{n-1}(x)D_x + a_n(x)]y = f(x) \quad (3.1.2)$$

in which $a_i(x)$, $(i = 0, \ldots, n)$ are polynomials.

The point $x = x_0$ is called an *ordinary point* of the equation if $a_0(x_0) \neq 0$.

Any point $x = x_1$ for which $a_0(x_1) = 0$ is called **singula**r *point* of the differential equation.

The point $x = x_1$ is called *a regular point* if the equation (3.1.1) with $f(x) = 0$ can be written in the form

$$[(x-x_1)^n D^n + (x-x_1)^{n-1} b_1(x) D^{n-1} + (x-x_1) b_2(x) D^{n-2} + \ldots + (x-x_1) b_{n-1}(x) D + b_n(x)]y = f(x) \quad (3.1.3)$$

where $b_i(x), (i = 1,\ldots,n)$ are analytic $x = x_1$.

Examples

List the singular points for:

(a) $(x-3)y^{//} + (x+1)y = 0$

[Solution: $x = 3$*]*

(b) $(x^2+1)y^{///} + y^{//} - x^2 y = 0$

[Solution: $x = \pm i$*]*

Learning Activity 3.1.2 List the singular points for:

(i) $8y^{///} - 3x^3 y^{//} + 4 = 0$ *.[Solution: None]*

(ii) $(x-1)^2 y^{//} - x(x-1)y^{/} + xy = 0$ *.[Solution:* $x = 1$ *regular]*

The expression, *"find a solution about the point* $x = x_0$ *,"* is used in discussing power series solutions of differential equations. It means to obtain a series in powers of

$(x = x_0)$

which is valid in a region (neighborhood) about the point x_0, and which is an expansion of a function $y(x)$ that will satisfy the differential equation.

3.2 Method of successive differentiation

This method is also called the Taylor's series method. It involves finding the power series solutions of the differential equation

$$p(x)y'' + q(x)y' + r(x)y = 0 \qquad (3.2.1)$$

where $p(x), q(x)$ and $r(x)$ are polynomials, about an ordinary point $x = a$.

On solving (3.2.1) for y'',we get

$$y'' = -\frac{q(x)y' + r(x)y}{p(x)} \qquad (3.2.2)$$

As we saw earlier, a value x which is such that $p(x) = 0$ is called *a singular point* or *singularity,* of the differential equation (3.2.1). Any other value of x is called an *ordinary point* or *non –singular point.*

The method uses the values of the derivatives evaluated at the ordinary point, which are obtained from the differential equation (3.2.1) by successive differentiation. When the derivatives are found, we then use Taylor's series

$$y(x) = y(a) + y'(a)(x - a_0) + \frac{y''(a)(x-a)^2}{2!} + \frac{y'''(a)(x-a)^3}{3!} + \ldots (3.2.3)$$

giving the required solution.

Example 3.2

Find the solution of $xy'' + x^3y' - 3y = 0$ that satisfies $y = 0$ and $y' = 2$ at $x = 1$

Solution

$$y'' = -x^2y' + 3x^{-1}y$$

$$y''' = -x^2y'' - (2x - 3x^{-1})y' - 3x^{-2}y,$$

$$y^{iv} = -x^2y''' - (4x - 3x^{-1})y'' - (2 + 6x^{-2})y' + 6x^{-3}y.$$

Evaluating these derivatives at $x = 1$

$$y''(1) = -2,$$

$$y'''(1) = 4,$$

$$y^{iv}(1) = -18.$$

Substituting in the Taylor's series (3.2.3), the solution is

$$y(x) = 0 + 2(x - 1) - \frac{2(x-1)^2}{2} + \frac{4(x-1)^3}{6} + \frac{18(x-1)^4}{24} + \ldots$$

$$= 2(x-1) - (x - 1)^2 + \frac{2(x-1)^3}{3} + \frac{3(x-1)^4}{4} + \ldots$$

Learning Activity 3.2

(i) Reading: Please read Mauch, S (2004,pp.1184-1198).

(ii) Group Discussion

Construct second order differential equations of the form (3.1.3).For each of the equations you construct, find the singularity, the ordinary point, and the regular point respectively.

(iii) Problem solving (Group Work)

(a) Solve $y' = x^2 + y^2$ given that $y = 1$ at $x = 0$.

[Solution $y(x) = 1 + x + \frac{2x^2}{2!} + \frac{8x^3}{3!} + \frac{28x^4}{4!} + \frac{144x^5}{5!} + \ldots$*]*

(b) Find the solution of $(x-1)y''' + y'' + (x-1)y' + y = 0$

[Solution: $y(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots = \sin x$]

3.3 Method of undetermined coefficients

If x_0 is an ordinary point of the given differential equation, the solution can be expanded in the form

$$y(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ldots + c_k(x - x_0)^k + \ldots = \sum_{i=0}^{\infty} c_i (x - x_0)^i$$
(3.3.1)

It remains to determine the coefficients c_i, $(i = 0, \ldots)$. We differentiate the series (3.3.1), term by term, to obtain

$$y'(x) = 2c_2(x - x_0) + c_3(x - x_0)^2 + \ldots = \sum_{i=0}^{\infty} i c_i (x - x_0)^{i-1}$$
(3.3.2)

These values are now substituted into the given differential equation and regrouped into the terms of $(x - x_0)^i$, i.e.

$$\sum_{i=0}^{\infty} C_i (x - x_0)^i = 0 \qquad (3.3.4)$$

where C_i are functions of c_i, and the equation is an identity in $(x - x_0)$ so that $C_i = 0, i = 0, \ldots$ This will determine the values of c_i.

Example 3.3

Find a series solution of

$$(x^2 - 1)y'' - 2xy' - 4y = 0 \qquad (3.3.5)$$

Solution:

Since we desire the solution of an ordinary point, we choose the simplest such point, which is $x = 0$ for this example. We then write

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots,$$

$$y'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + 6c_6 x^5 + \ldots, (3.3.6)$$

$$y''(x) = 2c_2 + 6c_3 x + 12c_4 x + 20c_5 x^3 + 30c_6 x^4 + \ldots,$$

Equations (3.3.6) are now substituted in equation (3.3.5) and the like terms in x are collected .It is recommended that you do this in a tabular form as follows:

	x^0	x^1	x^2	x^3	x^4	x^5
$x^2 y''$			$2c_2$	$6c_3$	$12c_4$	…
$-y''$	$-2c_2$	$-6c_3$	$-12c_4$	$-20c_5$	$-30c_6$	…
$-2xy'$		$-2c_1$	$-4c_2$	$-6c_3$	$-8c_4$	…
$-4y$	$-4c_0$	$-4c_1$	$-4c_2$	$-4c_3$	$-4c_4$	…
Sum	0	0	0	0	0	…

Adding the coefficients column wise:

$2c_2 + 4c_0 = 0$; $\quad c_2 = -2c_0$

$6c_3 + 6c_1 = 0$; $\quad c_3 = -c_1$

$12c_4 + 6c_2 = 0$; $\quad c_4 = -\frac{1}{2}c_2 = c_0$

$20c_5 + 4c_3 = 0$; $\quad c_5 = -\frac{1}{5}c_3 = \frac{1}{5}c_1$

$30c_0 + 0c_4 = 0$; $\quad c_6 = 0$

The first few terms of the series can now be written in the form

$$y(x) = c_0 + c_1x - 2c_0x^2 - c_1x^3 + c_0x^4 + \frac{1}{5}c_1 + 0x^6$$

$$= c_0(1 - 2x^2 + x^4 + ...) + c_1(x - x^3 + \frac{1}{5}x^5 + ...)$$

Learning Activity 3.3

(i) Problem solving:

First try this problem on your own using example (3.3) as support material.

Find the series solution of:

(a) $(2x-1)y'' - 3y' = 0$

(b) $(2x^2+1)y'' + 3xy' - 6y = 0$

(ii) Group Discussion:

Discuss your solutions in part (i) in a small group. Pay attention to how other

members of the group have solved the same problem. Ask them questions on how they arrived at their solution.

(iii) Further Reading:

http://www.answers.com/topic/power-series-method.

Wikipedia information about **power series method**

3.4 Special functions

In mathematics, **special functions** are particular functions such as the trigonometric functions that have useful properties, and which occur in different applications often enough to warrant a name and attention of their own. There are many viewpoints on special functions ranging from the classical theory, through the twentieth century to the contemporary theories on special functions. Some of the special functions include Bessel functions, Beta functions, Elliptic integral functions, Hyperbolic functions, parabolic cylinder functions, error functions, gamma functions, and Whittaker functions. The list is longer than this! More information on the theory of special function is available http://en.wikipedia.org/wiki/Special_functions.

Learning activity: Read the web page http://en.wikipedia.org/wiki/Special_functions and use the available links on this page to identify more special functions.

Learning Activity # 4

Partial differential equations; Laplace transforms, Fourier series, and their applications.

Specific learning Objectives

By the end of this unit, you should be able to:

- State what a partial differential equation is;
- Correctly define the terminologies associated with partial differential equations;
- Obtain solutions of some simple partial differential equations;
- Apply the method of variation of parameters to solve second order partial differential equations;
- Define Laplace transform and Fourier series respectively;
- Appreciate the application of Laplace transforms and Fourier series in solving physical problems e.g. in heat conduction.

Summary

Mathematical formulations of problems involving two or more independent variables lead to partial differential equations.In this learning activity partial differential equations (PDEs) and terminologies associated with PDEs are defined. Solutions of some simple PDEs are introduced. The method of variation of parameters is discussed with respect to second order PDEs. Laplace transforms and Fourier series are also defined, and their use in solving boundary value problems are discussed.

Compulsory reading (Core text)

The Core text for this activity is Mauch, S (2004). *Introduction to Methods of Applied Mathematics* : Mauch Publishing Company.This is available on the course CD.

Additional general reading

Wikibooks, *Differential Equations*

Key Words

Independent variables: The expression $y = 3x^2 + 7$ defines y as a function of x when it is specified that the domain is (for example) is a set of real numbers; y is then a function of x ,a value of y is associated with each real-number value of x by multiplying the square of x by 3 and adding 7,and x is said to be **independent variable** of the function y .

Partial differential equations:A partial differential equation (PDEs) is an equation involving more than one independent variable and partial derivatives with respect to these variables.

Variation of parameters (Refer to key words in learning activity # 3)

Laplace transform: The function f is the Laplace transform of g if

$$f(x) = \int e^{-xt} g(t)dt$$

where the path of integration is some curve in the complex plane. The custom is to restrict the path of integration to the real axis from 0 to $+\infty$ .Hence the formal expression for the Laplace transform is $f(x) = \int_0^\infty e^{-xt} g(t)dt$

Fourier series: A series of the form

$$\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + ... = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

for which there is a function f such that

for $n \geq 0$, $a_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$ and

for $n \geq 1$, $b_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

Boundary value problems: The problem of finding a solution to a given differential equation which will meet certain specified requirements for a given set of values.

Learning activitities

4.1 Partial differential equations (PDE) of second order

4.1.1 Introduction

In the preceding chapters, we were concerned with ordinary differential equations involving derivatives of one or more dependent variables with respect to a single independent variable. We learned how such differential equations arise, methods by which their solutions can be obtained, both exact and approximate, and considered applications to various scientific fields.

Mathematical formulations of problems involving two or more independent variables lead to partial differential equations. As you will expect, the introduction of more independent variables makes the subject of partial differential equations more complicated than ordinary differential equations.In the following discussion we limit ourselves to second order partial differential equations and to the method of separation of variables.

4.1.2 Some Definitions

4.1.2a ***A partial differential equation*** is an equation containing an unknown function of two or more variables and its partial derivatives with respect to these variables.

4.1.2b The ***order*** of a partial differential equation is that of the highest ordered derivative present.

Example 4.1.2b. $\frac{\partial^2 u}{\partial x \partial y} = 2x - y$ is a partial differential equation of order two, or a second order differential equation. The dependent variable is u; the independent variables are x and y.

4.1.2c The ***solution*** of a partial differential equation is any function which satisfies the equation identically.

4.1.2d The ***general solution*** is a solution which contains a number of arbitrary independent functions equal to the order of the equation.

4.1.2e A ***particular solution*** is one which can be obtained from the general solution by particular choice of the arbitrary functions.

Example 4.1.2e. As seen by substitution $u = x^2 y - \frac{1}{2}xy^2 + F(x) + G(y)$ is the partial differential equation. Because it contains two arbitrary independent functions $F(x)$ and $G(x)$, it is also the *general solution*. If in particular

$F(x) = 2\sin x$, $G(x) = 4y^4 - 5$ we obtain the *particular solution*,

$u(x, y) = x^2 y - \frac{1}{2}xy^2 + 2\sin x + 3y^4 - 5$

The class of differential equations which contain the partial derivatives with respect to a single variable is solved by ordinary differential equation techniques.

4.1.2f A ***singular solution*** is one which cannot be obtained from the general solution by particular choice of the arbitrary functions.

4.1.2g A ***boundary-value problem*** involving a partial differential equation seeks all solutions of a partial differential equation which satisfy conditions called *boundary conditions.*

4.2 Solutions of some simple partial differential equations

The class of partial differential equations containing partial derivatives with respect to a single variable may be solved by ordinary differential equations techniques.

In order to get some ideas concerning the nature of solutions of partial differential equations, let us consider the following problem for discussion.

4.2.1 Example: Obtain solutions of the PDE

$$\frac{\partial^2 U}{\partial x \partial y} = 6x + 12y^2 \qquad (4.2.1a)$$

Here the dependent variable U depends on two independent variables x and y. To find the solution we seek to determine $U(x, y)$ .i.e. U in terms of x and y If we write (4.2.1a) as

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right) = 6x + 12y^2 \qquad (4.2.1b)$$

We can integrate with respect to x keeping y constant, to find

$$\left(\frac{\partial U}{\partial y}\right) = 3x^2 + 12xy^2 + F(y) \qquad (4.2.1c)$$

where we have added the arbitrary "constant" of integration which can depend on y

denoted by $F(y)$ .We now integrate (4.2.1c) with respect to y keeping x constant getting

$$U = 3x^2 y + 4xy^3 + \int F(y)dy + G(x) \qquad (4.2.1d)$$

This time an arbitrary function of x , $G(x)$ is added .Since the integral of an arbitrary function of y is another arbitrary function of y , we can write (4.2.1d) as

$$U = 3x^2 y + 4xy^3 + H(y) + G(x) \qquad (4.2.1e)$$

This can be checked by substituting it back into (4.2.1a) and obtaining the identity. Equation (4.2.1e) is called the *general solution* of (4.2.1a).If $H(y)$ and $G(x)$ are known, e.g. $H(y) = y^3$ and $G(x) = \sin x$, equation (4.2.1a) is caled *a particular solution*.

In general given nth –order partial differential equation, a solution containing n arbitrary functions is called the *general solution*, and any solution obtained from this general solution by particular choices of the arbitrary constant is called particular solution.

As in the case of ordinary differential equations, it may happen that there are singular solutions which cannot be obtained from the general solution by any choice of the arbitrary functions. For example, suppose we want to solve (4.2.1a) subject to two conditions

$$U(1, y) = y^2 - 2y , \; U(x,2) = 5x - 5 \qquad (4.2.1f)$$

Then from the general solution (4.2.1e),and the first condition of (4.2.1f) we get

$$U(1, y) = 3(1)^2 y + 4(1) y^3 + H(y) + G(1) = y^2 - 2y$$

or $H(y) = y^2 - 5y - 4y^3 - G(1)$

so that $U = 3x^2y + 4xy^3 + y^2 - 5y - 4y^3 - G(1) + G(x)$ (4.2.1g)

If we now use the second condition in (4.2.1f) in the general solution (4.2.1e)

$U(x,2) = 3x^2(2) + 4x(2)^3 + (2)^2 - 5(2) - 4(2)^3 - G(1) + G(x) = 5x - 5$
(4.2.1h)

from which $G(x) = 33 - 27x - 6x^2 + G(1)$.

Using this in (4.2.1g), we obtain the required solution

$$U = 3x^2y + 4xy^3 + y^2 - 5y - 4y^3 - G(1) + 33 - 27x - 6x^2 + G(1)$$

$$U = 3x^2y + 4xy^3 + y^2 - 5y - 4y^3 - 27x - 6x^2 + 33 \quad (4.2.1i)$$

4.3 The Method of Separation of Variables

In this method it is assumed that a solution can be expressed as a product of unknown functions each of which depends on only one of the independent variables. The success of the method hinges on our being able to write the resulting equation so that one side depends only on one variable, while the other side depends on the remaining variables so that each side must be a constant. By repetition of this the unknown functions can be determined. Superposition of these solutions can then be used to find the actual solution.

Theorem 4.3.1

Let the linear partial differential equation

$$\phi(D_x, D_y, ...)U = F(x, y, ...) \quad (4.3.1a)$$

where $x, y, ...$ are independent variables and $\phi(D_x, D_y, ...)$ is a polynomial operator in $D_x, D_y, ...$.Then the general solution of (4.3.1a) is the sum of solution U_c of the complementary function $\phi(D_x, D_y, ...)U = 0$ (4.3.1b)

and any particular solution U_p of (4.3.1a) .

In short the general solution $U = U_c + U_p$ (4.3.1c)

Theorem 4.3.2

Let $U_1, U_2, \ldots$ be solutions of equations $\phi(D_x, D_y, \ldots)U = 0$.

Then if $a_1, a_2, \ldots$ are any constants

$$U = a_1U_1 + a_2U_2 + \ldots \qquad (4.3.2a)$$

is also a solution.

This theorem is referred to as the *principle of superposition.*

Assume a solution of (4.3.1b) to be of the form

$$U = X(x)Y(y) \text{ or briefly } U = XY \qquad (4.3.2b)$$

i.e. a function of x alone times a function of y.

The method of solution using (4.3.2b) is called the method of separation of variables (MSOV).The best way to illustrate MSOV is through an example.

Example 4.3.2.Solve the boundary value problem

$$\frac{\partial U}{\partial x} + 3\frac{\partial U}{\partial y} = 0 \; U(0, y) = 4e^{-2y} - 3e^{-6y} \qquad (4.3.2c)$$

Solution. Here the dependent variables are x and y, so we substitute $U = XY$ in the given differential equation where X depends on only x and Y depends on only y.

$$\frac{\partial(XY)}{\partial x} + 3\frac{\partial(XY)}{\partial y} = 0 \text{ or } X'Y = -3XY'$$

$$\frac{X'}{3X} = -\frac{Y'}{Y} \qquad (4.3.2d)$$

Since one side depends only on x and the other side depends only on y, and since X and Y are independent variables equation (4.3.2c) can be true if and only if each side is equal to the same constant.

$$\frac{X'}{3X} = -\frac{Y'}{Y} = c \qquad (4.3.2d)$$

From (4.3.2d) we have therefore

$$X' - 3cX = 0 \qquad Y' + cY = 0 \tag{4.3.2e}$$

From our knowledge of ordinary differential equations, (4.3.2e)
have solutions

$$X = b_1 e^{3cx}, Y = b_2 e^{-cy} \tag{4.3.2f}$$

Thus

$$U = XY = b_1 b_2 e^{c(3x-y)} = Be^{c(3x-y)} \tag{4.3.2g}$$

where $B = b_1 b_2$

If we now use the condition in (4.3.2g) in (4.3.2c) we have

$$B\ e^{-cy} = 4e^{-2y} - 3e^{-6y} \tag{4.3.2h}$$

Unfortunately (4.3.2h) cannot be true for any choice of B and c and it would seem as if the method fails!.The situation is saved by **using Theorem 4.3.2** on the *superposition of solutions*.From (4.3.2g) we see that

$U_1 = b_1 e^{c_1(3x-y)}$ and $U_2 = b_2 e^{c_2(3x-y)}$ are both solutions and so the general solution should be

$$U = b_1 e^{c_1(3x-y)} + b_2 e^{c_2(3x-y)} \tag{4.3.3h}$$

The boundary condition of (4.3.2h) now leads to

$$b_1 e^{-c_1 y)} + b_2 e^{-c_2 y} = 4e^{-2y} - 3e^{-6y}$$

which is satisfied if we choose $b_1 = 4$, $c_1 = 2$, $b_2 = -3$, $c_2 = 6$

This leads to the required solution of (4.3.2g) given by

$$U = 4e^{2(3x-y)} - 3e^{6(3x-y)}$$

Learning Tip

A student may wonder why we did not work the above problem by first finding the general solution and then getting the particular solution. The reason is that except in very simple cases, the solution is often difficult to find, and even when it can be obtained, it may be difficult to determine the particular solution from it.

However, experience shows that for more difficult problems which arise in practice, separation of variables combined with the principle of superposition prove to be successful.

Learning Activity 4.3

(A) Group Discussion.

Determine whether each of the following partial differential equations is linear or nonlinear. State the order of each equation, and name the dependent and independent variables. *Then check the solutions of these questions after your discussions. Do your solutions agree with the ones provided?*

CAUTION: Please check the solutions only after you have tried out the problems and noted down your response to each question.

(a) $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ [Solution: *linear; order2; dependent var.* u *; independent var.* $x, t.$*]*

(b) $x^2 \frac{\partial^3 R}{\partial y^3} = y^3 \frac{\partial^2 R}{\partial x^2}$ [Solution: *linear; order3; dependent var.* R *; independent var.* $x, y.$*]*

(c) $W \frac{\partial^2 W}{\partial r^2} = rst$ [Solution: non*linear; order2; dependent var.* W *; ind pendent var.* $r, s, t.$*]*

(d) $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$ [Solution: *linear; order2; dependent var.* φ *; independent var.* $x, y, z..$*]*

(e) $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 1$ [Solution: non*linear; order1; dependent var.* z *; independent var.* $u, v.$*]*

(B) Reading:

Please read Mauch, S (2004, pp.1704-1705) provided on your **CD.**

Use the examples to solve the following problems:

Obtain the solution of the following boundary value problem

(i) $$\frac{\partial^2 V}{\partial x \partial y} = 0; V(0,y) = 3\sin y,\ V_x(x,1) = x^2$$

(ii) $$\frac{\partial^2 U}{\partial x \partial y} = 4xy + e^x;\ U_y(0,y) = y, U(x,0) = 2.$$

Compare notes with other group members and discuss any variations in your approaches.

4.4 Laplace transforms

In the preceding discussion in this module you learned how to solve linear differential equations with constant coefficients subject to given conditions, called boundary or initial conditions. You will recall that the method used consists of finding the general solution of the equations in terms of the number of arbitrary constants and then determining these constants from the given conditions. In the course of solving problems of differential equations you must have met a number of challenges in getting to the solutions. You probably wished that there were other techniques you could use to solve such problems. The method of Laplace transforms offers one more powerful technique of solving problems of differential equations. This method has various advantages over other methods.

First by using the method, we can transform a given differential equation into an algebraic equation. Secondly, any given initial conditions are automatically incorporated into the algebraic problem so that no special consideration of them is needed. Finally use of tables of Laplace transforms does reduce the labour of obtaining solutions just as the use of tables of integrals reduces labour of integration.

Definition 4.4.1

The Laplace transform of a function $f(t)$ is defined as

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt \quad (4.4.1)$$

and is said to exist or not according as integral in (4.4.1) exists [converges] or does not exist [diverges].The set of values $s > s_0$ ($s_0 \in R$) for which (4.4.1)exists is called the *range of convergence or existence* of $L\{f(t)\}$. It may happen however that (4.4.1) does not exist for any value of s. The symbol L in (4.4.1) is called the Laplace transform operator.

The Laplace transforms of some elementary functions are given in the following table for your easy reference

	$f(t)$	$L\{f(t)\} = F(s)$
1.	1	$\frac{1}{s}$ $\quad s > 0$
2.	$t^n \quad n = 1,2,3,...$	$\frac{n!}{s^{n+1}}$ $\quad s > 0$
3.	$t^p \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$ $\quad s > 0$
4.	e^{at}	$\frac{1}{s-a}$ $\quad s > a$
5.	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$ $\quad s > 0$
6.	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$ $\quad s > 0$
7.	$\cosh at$	$\frac{a}{s^2-a^2}$ $\quad s > \lvert a \rvert$
8.	$\sinh at$	$\frac{s}{s^2-a^2}$ $\quad s > \lvert a \rvert$

Example. 4.4.2 Solve $y'' - 3y' + 2y = 2e^{-t}$, $y(0) = 2,\ y'(0) = -1$

Solution. Taking the Laplace transform of the given differential equation,

$$[s^2Y - sy(0) - y'(0)] - 3[sY - y(0)] + 2Y = \frac{2}{s = 1}$$

Then using initial conditions $y(0) = 2,\ y'(0) = -1$ and solving the algebraic equation for Y, we find using partial fractions

$$Y = \frac{2s^2 - 5s - 5}{(s+1)(s-1)(s-2)} = \frac{1/3}{(s+1)} + \frac{4}{(s-1)} + \frac{-7/3}{(s-2)}$$

Taking the inverse Laplace transform , we obtain the require solution.

$$y = \frac{1}{3}e^{-t} + 4e^{t} - \frac{7}{3}e^{2t}$$

Learning Activity 4.4

Reading: Please read Mauch, S (2004, pp.1475-1492) provided on your **CD**

Problem solving: Use the reading and the notes to work out the following problems;

Solve the by the method of Laplace transform

(i) $y''(t) + y'(t) = 1,\ y(0) = 1,\ y'(0) = 0$

(ii) $y''(t) - 3y'(t) + 2y(t) = 4,\ y(0) = 1,\ y'(0) = 0$

4.5 Fourier Series

Fourier series is named after the man who discovered it in his researches on heat flow involving partial differential equations. The series has many applications in Physical problems. For example in the heat conduction (and diffusion)equation

$$\frac{\partial U}{\partial t} = \frac{1}{k}\frac{\partial^2 U}{\partial x^2}$$

where k constant and $U(x,t)$ is the temperature at place x at time t.

Definition 4.5.1

Given a function $f(x)$ defined in the interval $-L \le x \le L$,evaluate the coefficients called Fourier coefficients , given by

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}dx \ , \ b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}dx \ (4.5.1a)$$

Using those coefficients in (4.3), and assuming the series converges to $f(x)$,the required Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}\right) \qquad (4.5.1b)$$

4.5.2 Application to heat conduction equation

Example: A metal bar 100cm long has ends $x = 0$ and $x = 100$ kept at 0^0C. Initially, half of the bar is at 60^0, while the other half is at 40^0 C Assuming a heat conduction coefficient of 0.16 units, and that the surface of the bar is insulated, find the temperature everywhere in the bar at time t.

Mathematical formulation

The heat conduction equation is

$$\frac{\partial U}{\partial t} = 0.16\frac{\partial^2 U}{\partial x^2} \qquad (4.5.2a)$$

where $U(x,t)$ is the temperature at place x at time t.the boundary conditions are

$$U(0,t) = 0 \, , \, U(100,t) = 0 \ , \ U(x,0) = \begin{cases} 60, & 0 < x < 50 \\ 40, & 50 < x < 100 \end{cases} \qquad (4.5.2b)$$

Solution

Assuming $U = XT$ in (4.5.2a)and from previous discussion in section 4.3 on partial differential, we get

$$XT' = 0.16X''T \quad \text{or} \quad \frac{T'}{0.16T} = \frac{X''}{X} \qquad (4.5.2c)$$

Setting these equal to a constant , which as our previous experience indicated was a negative, and which we denote by $-\psi^2$, we have

$$\frac{T^{/}}{0.16T} = \frac{X^{//}}{X} = -\psi^2$$

or $\quad T^{/} + \psi^2 T = 0 \quad , \; X^{//} + \psi^2 X = 0$ (4.5.2d)

The solution to (4.5.2.d) is

$$U(x,t) = e^{-0.16\psi^2 t}(A\cos\psi x + B\sin\psi x) \qquad (4.5.2e)$$

The first two conditions in (4.5.1b) show that $A = 0, \psi = n\pi/100$.

To satisfy the last condition of (4.5.1b), we use superposition of the solutions to obtain

$$U(x,t) = b_1 e^{-16(10)^{-6}\pi^2 t}\sin\frac{\pi x}{100} + b_2 e^{-64(10)^{-6}\pi^2 t}\sin\frac{2\pi x}{100} + \ldots \quad (4.5.2f)$$

For $t = 0$,

$$b_1 \sin\frac{\pi x}{100} + b_2 \sin\frac{2\pi x}{100} = U(x,0) \qquad (4.5.1g)$$

Thus using (4.3a),we have

$$b_n = \frac{2}{100}\int_0^{100} U(x,0)\sin\frac{n\pi x}{100}dx$$

$$= \frac{2}{100}\int_0^{50}(60)\sin\frac{n\pi x}{100}dx \; b_n + \frac{2}{100}\int_{50}^{100}(40)\sin\frac{n\pi x}{100}dx$$

$$= \frac{120}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) + \frac{80}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

Thus $b_1 = \frac{200}{\pi}$, $b_2 = \frac{40}{\pi}$ ... and equation (4.5.1f) becomes

$$U(x,t) = \frac{200}{\pi} e^{-16(10)^{-6}\pi^2 t} \sin\frac{\pi x}{100} + \frac{40}{\pi} e^{-64(10)^{-6}\pi^2 t} \sin\frac{2\pi x}{100} + \ldots$$

which is the required solution.

4.5.3 Spherical harmonics (SH)

The **spherical harmonics** are the angular portion of an orthogonal set of solutions to Lap lace's equation, represented in a system of spherical coordinates. Spherical harmonics have many useful applications in mathematics and the physical sciences. In general, the treatment of spherical harmonics involves some mathematical rigour. For purposes of this introductory course however; we will limit ourselves to the theoretical and practical applications spherical harmonics.

Reading: http://en.wikipedia.org/wiki/Spherical_harmonic

Read the above webpage and note some applications of spherical harmonics in Mathematics, Physics, Chemistry, Biology, and ICT. Note particularly the 3D graphics application, also discussed in ICT Basic Skills Module 4.

Learning Activity 4.5

Reading

Please read Mauch, S. (2004, pp.1333-1345) provided on your CD

Group Work

Discuss the solution to the following problem in a small group of 5-6 members.

Learning Tip

Use Example 4.5.2 and the general principle of separation of variables and superposition, to discuss the solution of the(heat conduction) problem:

It is important that you set up an appropriate model of the partial differential equation to be solved .The model equation should incorporate all the variables given.

A metal bar 100cm long has ends $x = 0$ and $x = 100$ kept at 0^0C. Initially, the right half of the bar is at 0^0C, while the other half is at 80^0 C. Assuming a heat conduction (diffusion) coefficient of 0.20 units, and an insulated surface, find the temperature at any position of the bar at any time.

XI. Glossary of Key Concepts

Boundary value problems:The problem of finding a solution to a given differential equation which will meet certain specified requirements for a given set of values.

Degree: The degree of the an ordinary differential equation is power to which the highest ordered derivative in the given equation is raised

Differential equation: A differential equation is a relation between a function and its derivatives

Fourier series: A series of the form

$$\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + ... = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

, for which there is a function f such that

for $n \geq 0$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$ and

for $n \geq 1$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

Independent variables: The expression $y = 3x^2 + 7$ defines y as a function of x when it is specified that the domain is (for example) is a set of real numbers; y is then a function of x ,a value of y is associated with each real-number value of x by multiplying the square of x by 3 and adding 7,and x is said to be independent variable of the function y .

Inverse technique: This technique is applied in solving differential equation using the properties of a differential operator

Laplace transform: The function f is the Laplace transform of g if

$$f(x) = \int e^{-xt} g(t)dt$$

where the path of integration is some curve in the complex plane. The custom is to restrict the path of integration to the real axis from 0 to $+\infty$ .Hence the formal expression for the Laplace transform is $f(x) = \int_0^{\infty} e^{-xt} g(t)dt$

Non-homogeneous equations: An homogeneous differential equation is one with the right hand side equated to zero. A non- homogeneous differential equation is an equation with the right hand side not equal to zero.

Order: The order of a DE is an integer showing of the highest ordered derivative a given equation.

Partial differential equations:A partial differential equation (PDEs) is an equation involving more than one independent variable and partial derivatives with respect to these variables.

Power series: A series whose terms contain ascending positive integral powers

of a variable. e.g. $a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$, where the $a's$ are constants and x is a variable.

Undetermined coefficients: These are constants to be explicitly determined by solving the particular integral of a differential equation. The method that does this is called the method of undetermined coefficients

Variable coefficients: Unlike differential equations with constant coefficients, there also differential equations with variable coefficients. Essentially, a variable coeffiecient is one that is not constant i.e. it is expressed in functional form

Variation of parameters: This is a method of finding a particular solution of a linear differential equation when the general solution of the reduced equation(homogeneous equation)is known. (See notes below for details)

Taylor's series: In general if any function can be expressed as a power series such

as $c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + ... + c_n(x-a)^n + ...$, that series is a Taylor's series.

Successive differentiation: One of the methods of finding the power series solutions of a differential equation.

XII. Compiled List of Compulsory Readings

Mauch, S.(2004).*Introduction to Methods of Applied Differential Equations*. Mauch Publishing Company.

Required chapters: 14, 15, 16, 17, 21, 23, 28, 30, 35, 36, 37.

Abstract

This book has a lot of material in mathematics all compiled in one volume.It is written in such away that it will support self –study as well as conventional learning.The material is presented in an-easy- to- follow- sequence.The book treats Differential equation in great detail and provides examples as well as worked out solutions to the Exercises.One of the major advantages of the book is its availability online free of charge. It is also downloadable on CD. In that respect, distant learners particularly those from the poor rural settings in Africa will benefit much from the free access to the book on CD.

XIII. Compiled List of (Optional) Multimedia Resources

wxMaxima is required for this course. It is a computer algebra system (CAS) which is a software programme. You are required to install this onto a computer where you can use the software to explore differential equations. The software is open-source which means that it is completely free for you to use. The installation files are included on the course CD. However, it is also available on the internet at: http:// wxmaxima.sourceforge.net/.

XIV. Compiled List of Useful Links

Abstract:

The following web pages are from the Wikibooks.They provide useful links to study materials on a topics in differential equations free of charge.They are therefore strongly recommended as supplementary material to the compulsory reading .The topics coverd are mentioned for each web page to give an indication of content.

Differential equations / First order

http://en.wikibooks.org/wiki/ Differntial _Equations/First_Order

Differential equations / separable variables

http://en.wikibooks.org/wiki/ Differntial _Equations/Separable_1

Differential equations /Exact1

http://en.wikibooks.org/wiki/ Differntial _Equations/ Exact_1

Differential equations / Substitution methods

http://en.wikibooks.org/wiki/ Differntial _Equations/Substitutions_1

Differential equations / Homogeneous 1

http://en.wikibooks.org/wiki/ Differntial _Equations/ Homogeneous_ 1

Differential equations / Non-Homogeneous 1

http://en.wikibooks.org/wiki/ Differntial _Equations/ Non_Homogeneous_1_

Introduction to Partial Differential equations

http://en.wikibooks.org/ Partial_Differntial _Equations

Partial Differential equations/ TheLaplacian and Laplace's Equation

http://en.wikibooks.org/ Partial_Differntial _Equations/ The Laplacian_ and_ Laplace's Equation

Details and applications of Fourier Series

http://en.wikibooks.org/ Partial_Differntial _Equations/ Details _and_Applications_of_ Fourier Series

XV. Synthesis of the Module

In this module Ordinary differential equations and higher order differential equations are covered under four major learning activities. In unit one homogeneous and non-homogeneous ordinary differential equations are covered. Some of techniques discussed include variation of parameters, the method of undetermined coefficients and inverse operators respectively. In unit two, series solutions of differential equations are discussed. Also discussed in unit two are partial differential equations and solutions by separation of variables.Other topics discussed are the Laplace transforms, Fourier series, Fourier transforms,special functions and spherical harmonics,and their applications.

At the end of this module the student teacher should be able to demonstrate an understanding of differential equations,and how it comes about in theory and practice.Then the next important aspect of the course is how to solve the differential equations using a variety of methods.Secondly the student teacher should also demonstrate an understanding of the concepts and properties of special functions and spherical harmonics and their applications in mathematics, physical sciences and ICTrespectively.

XVI. Summative evaluation

Summative Evaluation **Time: 3 hrs**

Instructions

Answer all questions from section **A** and any TWO (2) questions from section **B**

Section A *(Answer all questions from this section)*

Q1. State the order and the degree of the following differential equations.

a) $\left(\frac{dy}{dx}\right)^2 = e^x$ (2 marks)

b) $\frac{d^2y}{dx^2} + xy = \sin x$ (2 marks)

c) $\frac{d^2y}{dx^2} + xy\frac{dy}{dx} + y = 2$ (2 marks)

d) $\left(y\frac{d^4y}{dx^4}\right)^2 + 3\cos x\frac{d^3y}{dx^3} + \frac{dy}{dx} = 0$ (2marks)

Q2. Obtain the differential equation associated with the following solution.

$y = Ax^2 + Bx + C$ (8marks)

Q.3 Solve the initial value problem.

$\frac{dy}{dx} = x^2 + 3x^2y$, given that $y(0) = 2$ (8marks)

Q4. Establish whether the following equations are exact or not.

a) $\left(3x^2 - 2y^2\right)dx + \left(1 - 4xy\right)dy = 0$ (4marks)

b) $(2x^3 + 3y)dx + (3x + y - 1)dy = 0$ (4marks)

Q5 Find the general solution of the homogeneous differential equation

$$\frac{dy}{dx} = x^2 + 3x^2 y,$$ (8marks)

Section B *(Answer any two questions from this section)*

Q. 6 Solve the different equation

$y'' - 5y + 6y = 2e^x$ by the methods of
variation of parameter (20 marks)

Q.7 A cup of tea at 90^0 C is placed on a dinning table at constant room temperature of 20^0 C.From experience, it is known that it takes 10 minutes for tea to cool from 90^0 C to 70^0 C.

What will the temperature of the tea be after 30 minutes. (20 marks)

Q.8 Solve the differential equation:

$y'' + x^2 y = 0$

By the Power Series method. (20 marks)

Q.9 Solve using Laplace Transforms:

$$y'' - 3y' + 2y = 12e^{4t},\ y(0) = 1, y'(0) = 0$$ (20 marks)

Solutions

Q1.

	Order	Degree	Comments
(a)	1	2	Highest order = 1
(b)	2	1	Highest order = 2
(c)	2	1	Highest order = 2
(d)	4	2	Highest order = 4

Q2.

$$y = Ax^2 + Bx + C$$
$$y' = 2Ax + B$$
$$y'' = 2A$$
$$y''' = 0$$

The differential equation is $y''' = 0$

Q3. $\dfrac{dy}{dx} = x^2 + 3x^2 y = x^2(1+3y)$

Using the method of seperation of variables:

$$\frac{dy}{(1+3y)} = x^2 dx$$

Integrating both sides:

$$\frac{1}{3}\ln(1+3y) = \frac{x^3}{3} + \ln C$$
$$\ln(1+3y) = x^3 + K \qquad (K \text{ is a constant})$$
$$\therefore (1+3y) = Ae^{x^3} \qquad (A = e^K)$$

At $x = 0, y = 2$

$$(1+6) = A$$
$$A = 7$$

So, the required solution is:

$(1+3y) = 7e^{x^3}$

Q4(a). The general solution $M(x,y)dx + N(x,y)dy = 0$
Is said to be exact if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$(3x^2 - 2y^2)dx + (1-4xy)dy = 0$

$M = 3x^2 - 2y^2$ and $N = (1-4xy)$

$$\frac{\partial M}{\partial y} = -4y, \frac{\partial N}{\partial x} = -4y$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -4y$

The given equation is exact.

Q4(b). $(2x^3 + 3y)dx + (3x + y - 1)dy = 0$

$M = 2x^3 + 3y$ and $N = (3x + y - 1)$

$$\frac{\partial M}{\partial y} = 3, \frac{\partial N}{\partial x} = 3$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3$
The given equation is exact.

Q5. $y'' + 4y' + 13y = 0$

Let $y = e^{mx}$

The auxiliary equation is:

$$m^2 + 4m + 13 = 0$$

$$m = \frac{-4 \pm \sqrt{16-52}}{2}$$

$$= \frac{-4 \pm 6i}{2}$$

$$= -2 \pm 3i \qquad \text{(complex roots)}$$

There are two roots: $-2+3i$, $-2-3i$

The general solution is:

$$y = e^{-2x}[A\cos 3x + B\sin 3x]$$

Q6. $$y'' - 5y' + 6y = 2e^x$$
$$y'' - 5y' + 6y = 0$$

The auxiliary equation is $m^2 - 5m + 6 = 0$

$\Rightarrow m = 2$ and $m = 3$

$\therefore$ The complementary function is $y_c = Ae^{2x} + Be^{3x}$

Let $y_1 = e^{2x}$, $y_2 = e^{3x}$

The particular integral y_p is given by:

$$y_p = v_1 y_1 + v_2 y_2$$

Where: $$v_1(x) = \int \frac{-g(x)y_2}{y_1 y_2' - y_2 y_1'}dx$$

$$v_2(x) = \int \frac{g(x)y_1}{y_1 y_2' - y_2 y_1'}dx$$

And $g(x) = 2e^x$

$$v_1 = \int \frac{-2e^x e^{3x}}{e^{5x}} dx$$
$$= -2\int e^{-x} dx$$

$$v_2 = \int \frac{2e^x e^{2x}}{e^{5x}} dx$$
$$= 2\int e^{-2x} dx$$

So, $y_p = 2e^x - e^x$

The general solution is:

$$y = y_c + y_p$$
$$= Ae^{2x} + Be^{3x} + e^x$$

Q7. Let $H(t)$ be the temperature of the tea at time t and H_0 the temperature of the room at time t.

Then:

$$\frac{dH}{dt} = -k(H - 20)$$
$$\Rightarrow (H - 20) = Ae^{-kt}$$

At $t = 0$, $H = 90$ $, \Rightarrow A = 70$

Similarly at At $t = 10$, $H = 70$ $, \Rightarrow k = \frac{1}{10} \ln\left(\frac{7}{5}\right)$

$$\therefore\ H(t) = 70e^{-\frac{1}{10}\ln\left(\frac{7}{5}\right)t}$$

After 15 minutes ($t = 15$),

$$H(t) = 70e^{-\frac{1}{10}\ln\left(\frac{7}{5}\right)x15} = 51.6^0C$$

The temperature of tea after 15 minutes will be approximately .

Q8. Let $y = \sum_{n=0}^{\infty} a_n x^n$

Then
$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting into $y'' + x^2 y = 0$

$$\sum_{n=2}^{\infty} (n(n-1) a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} a_n x^n$$

Putting the series under the same powers of *x*:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

Summing all of the series for *n*=2 gives:

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_{n-2} \right] x^n = 0$$

Equating coeeficients of powers of *x*:

$$x^0 : 2a_2 = 0 \Rightarrow a_2 = 0$$
$$x^1 : 6a_3 = 0 \Rightarrow a_3 = 0$$

For x^n we either obtain the recursive formula:

$$a_{n+2} = \frac{-1}{(n+2)(n+1)} a_{n-2}, \; n \geq 2$$

Thus for:

$$n = 2; \quad a_4 = \frac{-1}{12} a_0$$

$$n = 3; \quad a_5 = \frac{-1}{20} a_1$$

$$n = 4; \quad a_6 = \frac{-1}{30} a_2 = 0 \quad (a_2 = 0)$$

$$n = 5; \quad a_7 = \frac{-1}{42} a_3 = 0 \quad (a_3 = 0)$$

$$n = 6; \quad a_8 = \frac{-1}{56} a_4 = \frac{1}{56 \times 12} a_0$$

And:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 (1 - \frac{1}{12} x^4 + \frac{1}{672} x^8 - \ldots) + a_1 (x - \frac{1}{20} x^5 + \ldots)$$

Q9. The Laplace transformations of the given equations yields:

$$(s^2 \overline{Y} - sY_0 - Y_1) - 3(s\overline{Y} - Y_0) + 2\overline{Y} = \frac{12}{s-4}$$

Putting the initial conditions into the equation: $Y_0 = 1$ and $Y_1 = 0$

$$\overline{Y}(s^2 - 3s + 2) - s + 3 = \frac{12}{s-4}$$

Now: $$\overline{Y} = \frac{s^2 - 7s + 24}{(s-1)(s-2)(s-4)}$$

$$\frac{s^2 - 7s + 24}{(s-1)(s-2)(s-4)} \equiv \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-4)}$$

Solving the partial fractions we obtain: $A = 6, B = -7, C = 2$

$$\bar{Y} = \left\{ \frac{6}{s-1} - \frac{7}{s-2} + \frac{2}{s-4} \right\}$$

Thus:

$$\begin{aligned} y &= L^{-1}\left\{\bar{Y}\right\} \\ &= L^{-1}\left\{ \frac{6}{s-1} - \frac{7}{s-2} + \frac{2}{s-4} \right\} \\ &= 6e^{t} - 7e^{2t} + 2e^{4t} \end{aligned}$$

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XVIII. Student Records

Name of the EXCEL file: Student_ Assessment Record_ Differential_ Equation

Student name	Registration No.	Assessment 1	Assessment 2	Assessment 3	Average score	Final Grad	Remark

XIX. Main Author of the Module

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George Ekol is Lecturer and Acting Head, Department of Mathematics, Kyambogo University.He has a Master of Science degree in Mathematics from Makerere University, Uganda.His research interest include statistical modelling , statistical computing & data analysis, statistics and mathematics education.He has presented a numbers of research papers in both local and international Conferences including ICME-9, Tokyo (2000);ICSTME, Goa, India(2001);ASE, UK(2002);ICMI-Study 14, Dortmund, Germany (2004);ICME-10, Denmark(2004);ICMI-Regional Conference, Johannesberg, South Africa(2005); and Park City Mathematics Institute, Utah,USA(2005,2006).

He is a member of Uganda Mathematical society(UMS),Uganda National Academy of Sciences, (UNAS),International Association of Statistics Education (IASE),International Association of Statistical Computing (IASC), and International Statistical Institute(ISI).

He was involved in the design of Basic ICT course with the African Virtual University (AVU) in 2005.