NOTICE

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I. Mathematics 1, Basic Mathematics

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II. Prerequisite Courses or Knowledge

Unit 1: (i) Sets and Functions (ii) Composite Functions

Secondary school mathematics is prerequisite.
This is a level 1 course.

Unit 2: Binary Operations

Basic Mathematics 1 is prerequisite.
This is a level 1 course.

Unit 3: Groups, Subgroups and Homomorphism

Basic Mathematics 2 is prerequisite.
This is a level 2 course.

III. Time

120 hours

IV. Material

The course materials for this module consist of:
Study materials (print, CD, on-line)
(pre-assessment materials contained within the study materials)
Two formative assessment activities per unit (always available but with specified submission date). (CD, on-line)
References and Readings from open-source sources (CD, on-line)
ICT Activity files
Those which rely on copyright software
Those which rely on open source software
Those which stand alone
Video files
Audio files (with tape version)
Open source software installation files
Graphical calculators and licenced software where available
V. Module Rationale

The rationale of teaching Basic mathematics is that it plays the role of filling up gaps that the student teacher could be having from secondary school mathematics. For instance, a lack of a proper grasp of the real number system and elementary functions etc. It also serves as the launching pad to University Mathematics by introducing the learner to the science of reasoning called logic and other related topics.
VI. Content

6.1 Overview

This module consists of three units which are as follows:

**Unit 1: (i) Sets and Functions (ii) Composite Functions**

This unit starts with the concept of a set. It then introduces logic which gives the learner techniques for distinguishing between correct and incorrect arguments using propositions and their connectives. A grasp of sets of real numbers on which we define elementary functions is essential. The need to have pictorial representations of a function necessitates the study of its graph. Note that the concept of a function can also be viewed as an instruction to be carried out on a set of objects. This necessitates the study of arrangements of objects in a certain order, called permutations and combinations.

**Unit 2: Binary Operations**

In this unit we look at the concept of binary operations. This leads to the study of elementary properties of integers such as congruence. The introduction to algebraic structures is simply what we require to pave the way for unit 3.

**Unit 3: Groups, Subgroups and Homomorphism**

This unit is devoted to the study of groups and rings. These are essentially sets of numbers or objects which satisfy some given axioms. The concepts of subgroup and subring are also important to study here. For the sake of looking at cases of fewer axiomatic demands we will also study the concepts of homomorphisms and isomorphisms. Here we will be reflecting on the concept of a mapping or a function from either one group to the other or from one ring to the other in order to find out what properties such a function has.
6.2 Outline

Unit 1: (i) Sets and Functions (ii) Composite Functions (50 hours)

Level 1. Priority A. No prerequisite.
Sets (4)
Elementary logic (8)
Number systems (6)
Complex numbers (4)
Relations and functions (8)
Elementary functions and their graphs (8)
Permutations (7)
Combinations (5)

Unit 2: Binary Operations (35 hours)

Level 1. Priority A. Basic Mathematics 1 is prerequisite.
Binary operations. (7)
Elementary properties of integers. (7)
Congruence. (7)
Introduction to Algebraic structures. (7)
Applications (7)

Unit 3: Groups, Subgroups and Homomorphism (35 hours)

Level 2. Priority B. Basic Mathematics 2 is prerequisite.
Groups and subgroups. (7)
Cyclic groups. (2)
Permutation groups. (5)
Group homomorphisms. (4)
Factor groups. (3)
Automorphisms. (3)
Rings, sub-rings, ideals and quotient rings. (7)
Isomorphisms theorems for groups and rings. (4)
This diagram shows how the different sections of this module relate to each other.

The central or core concept is in the centre of the diagram. (Shown in red).

Concepts that depend on each other are shown by a line.

For example: Set is the central concept. The Real Number System depends on the idea of a set. The Complex Number System depend on the Real Number System.

- Propositional Logic
- Real Number System
- Complex number system
- Homomorphisms and Isomorphisms
- Sets
- Functions and their graphs
- Trigonometry
- Groups and Rings
- Algebraic Structure
- Binary Operation
- Permutations and combinations
VII. General Objective(s)

You will be equipped with knowledge of elementary mathematical logic, sets, numbers and algebraic structures required for effective teaching of mathematics in secondary schools.

VIII. Specific Learning Objectives (Instructional Objectives)

By the end of this module, the learner should be able to…”

• Construct mathematical arguments.
• make connections and communicate mathematical ideas effectively and economically.
• Examine patterns, make abstractions and generalize.
• Understand various mathematical structures and the similarities and differences among these structures.
IX. Teaching and Learning Activities

Module 1: Basic Mathematics, Pre-assessment

Unit 1: Sets and Functions

Assessments and Solutions

Pre-assessment Questions

1. Given the quadratic equation:
   \[ 2x^2 - x - 6 = 0 \]
   The roots are
   a. \( \{ -4, 3 \} \)
   b. \( \{ 4, -3 \} \)
   c. \( \{ 2, -\frac{3}{2} \} \)
   d. \( \{ -2, \frac{3}{2} \} \)

2. The value of the function \( f(x) = 2x^2 + 3x + 1 \) at \( x = 3 \) is
   a. 19
   b. 28
   c. 46
   d. 16

3. Which of the following diagrams below represents the graph of \( y=3x(2-x) \)
4. The solution of the equation

\[ \sin x = -\frac{1}{2} \text{ in the range } 0 \leq x^\circ \leq 360 \text{ is:} \]

a. \( \{150^\circ, 210^\circ\} \)
b. \( \{30^\circ, 150^\circ\} \)
c. \( \{210^\circ, 330^\circ\} \)
d. \( \{30^\circ, 330^\circ\} \)

5. Given the triangle \( ABC \) below
Which of the following statements is correct?

a) \( \cos \alpha = \frac{2}{\sqrt{15}} \)
b) \( \sin \alpha = \frac{\sqrt{5}}{2} \)
c) \( \tan \alpha = 2 \)
d) \( \sec \alpha = \frac{1}{\sqrt{5}} \)

Unit 1: Pre-assessment Solutions

The following are the answers to the multiple choice questions.

Q 1 c  Q 2 b  Q 3 b  Q 4 c  Q 5 c

Unit 2: Binary Operations

1. The inverse of the function

\[ f(x) = \frac{1}{x - 1} \]

is

(a) \( f^{-1}(x) = x - 1 \)

(b) \( f^{-1}(x) = \frac{1 - x}{x} \)

(c) \( f^{-1}(x) = \frac{x + 1}{x} \)

(d) \( f^{-1}(x) = \frac{1}{x} - 1 \)
2. If $\sin \frac{x}{2} = \frac{a}{2}$ then

$\sin x$ in terms of $a$ is:

(a) $\frac{a}{\sqrt{4 - a^2}}$

(b) $a \sqrt{4 - a^2}$

(c) $a$

(d) $\frac{\sqrt{4 - a^2}}{2}$

3. A girl has 3 skirts, 5 blouses and 4 scarves. The number of different outfits consisting of skirt, blouse and scarf that she can make out of these is:

a. 220
b. 60
c. 12
d. 150

4. Given the complex number $z = 1 - i$ we have that $\text{Arg } z$ is:

(a) $45^\circ$

(b) $135^\circ$

(c) $225^\circ$

(d) $315^\circ$
5. If \( a * b = a^2 + ab - 1 \), then

\( 5 * 3 \) is

(a) 39
(b) 41
(c) 23
(d) 25

**Unit 2: Pre-assessment Solutions**

Q1. c  Q2. b  Q3. b  Q4. b  Q5. a

**Unit 3: Groups, Subgroups and Homomorphism**

1. Which of the following is a binary operation?
   (a) Squaring a number.
   (b) Taking the predecessor of a natural number.
   (c) Taking the successor of a natural number.
   (d) Finding the sum of two natural numbers

2. Recall the definition of a homomorphism and state which one of the following is a homomorphism on a group \( G \) of real numbers under either multiplication or addition?

   (a) \( \phi(x) = 2^x \)
   (b) \( \phi(x) = 6x \)
   (c) \( \phi(x) = x^2 \)
   (d) \( \phi(x) = x + 5 \)

3. For a group \( G \) if \( a * a = b \) in \( G \), then \( x \) is

   (a) \( b \)
   (b) \( ba^{-1} \)
   (c) \( a^{-1}b \)
   (d) \( a^{-1}ba^{-1} \)
4. If an element $a$ is a ring $\mathcal{R}$ is such that $a^2 = a$ then $a$ is called
   (a) nilpotent
   (b) characteristic
   (c) idempotent
   (d) identity

5. Let $\mathcal{R}$ be a ring and $x \in R$ if there exists a unique element $a \in \mathcal{R}$ such that $xa = x$, then $a$ is:
   (a) $e$
   (b) $a$
   (c) $-x$
   (d) $x$

**Unit 3: Pre-assessment Solutions**

1. d  2. c  3. d  4. c  5. d
Title of Pre-assessment: Pedagogical comment for learners

The questions in this pre-assessment are designed to test your readiness for studying the module.

The 5 questions preparing you for unit 1 require high school mathematics. If you make any errors, this should suggest the need to re-visit the high school mathematical topic referred to in the question.

The questions for unit 2 and unit 3 test your readiness after having completed the learning activities for unit 1 and unit 2.

If you make errors in the unit 2 pre-assessment, you should check through your work on unit 1 in this module. Likewise, if you make errors in the unit 3 pre-assessment, you should check through your work on unit 2 in this module.
X. Key concepts (glossary)

1. **Abelian group**: This is a group $\langle G, * \rangle$ in which $a * b = b * a$ for $a, b \in G$.

2. **Algebraic structure**: This is the collection of a given set $G$ together with a binary operation $*$ that satisfies a given set of axioms.

3. **Binary operation**: This is a mapping which assigns to each ordered pair of elements of a set $G$, exactly one element of $G$.

4. **Composite Function**: This is a function obtained by combing two or more other simple functions in a given order.

5. **Function**: This is a special type of mapping where an object is mapped to a unique image.

6. **Group**: This is a non-empty set say $G$ with a binary operation $*$ such that:
   
   (i) $a * b \in G$ for all $a, b \in G$.
   
   (ii) $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$.
   
   (iii) There exists an element $e$ in $G$ such that $e * a = a = a * e$ for all $a \in G$ where $e$ is called identity.
   
   (iv) For every $a \in G$ there exists
   
   $a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1} * a$
   
   Where $a^{-1}$ is called the inverse of $a$.

7. **Homomorphism**: This is a mapping $\phi$ from a group $G$ into another group $H$ such that for any pair $a, b \in G$. We have $\phi (ab) = \phi (a) \phi (b)$.

8. **Isomorphism**: This is a homorphism which is also a bijection.

9. **Mapping**: This is simply a relationship between any two given sets.
10. **Proposition**: This is a statement with truth value. Thus we can tell whether it is true or false

11. **Ring**: This is a non-empty set say $R$ with two binary operations $+$ and $*$ called addition and multiplication respectively such that:

   (i) $\langle R, + \rangle$ is an Abelian group.
   (ii) $\langle R, * \rangle$ is a multiplicative semigroup.

12. **Semigroup**: This is a non-empty set $S$ with a binary operation $*$ such that:

   (i) $a * b \in S$ for all $a, b \in S$.
   (ii) $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$.
   (iii) For all $a, b, c \in S$ we have:

   $$a * (b + c) = a * b + a * c \quad \text{and} \quad (a + b) * c = a * c + b * c$$

13. **Set**: This is a collection of objects or items with same properties

14. **Subgroup**: This is a subset $H$ of a group $G$ such that $H$ is also a group with respect to the binary operation in $G$. 
XI. Compulsory Readings

Reading #1
A Textbook for High School Students Studying Maths by the Free High School Science Texts authors, 2005, pg 38-47 (File name on CD: Secondary_School_Maths)

Reading #2
Elements of Abstract and Linear Algebra by E. H. Connell, 1999, University of Miami, pg. 1-13 (File name on CD: Abstract_and_linear_algebra_Connell)

Reading #3
Sets relations and functions by Ivo Duntsch and Gunther Gediga methodos publishers (UK) 2000. (File name on CD: Sets_Relations_Functions_Duntsch)

Reading #4
Abstract Algebra: The Basic Graduate Year, by Robert B. Ash (Folder on CD: Abstract_Algebra_Ash)

General Abstract and Rationale
All of the compulsory readings are complete open source textbooks. Together they provide more than enough material to support the course. However, the text contains specific page references to activities, readings and exercises which are referenced in the learning activities.
XII. Compulsory Resources

Wolfram MathWorld (visited 29.08.06)
http://mathworld.wolfram.com/

- A complete and comprehensive guide to all topics in mathematics. The students is expected to become familiar with this web site and to follow up key words and module topics at the site.

Wikipedia (visited 29.08.06)
http://www.wikipedia.org/

- Wikipedia provides encyclopaedic coverage of all mathematical topics. Students should follow up key words by searching at wikipedia.
### XIII. Useful Links

**Set Theory (visited 29.08.06)**

[http://www.mathresource.iitb.ac.in/project/indexproject.html](http://www.mathresource.iitb.ac.in/project/indexproject.html)

- Read through any of the sections by clicking on the slices of the pie.
- Especially work through the section called ‘functions’.
- Click the NEXT link at the bottom of the page to move forward.
- Click on double arrow ➔ buttons to see things move!

**Wolfram MathWorld (visited 29.08.06)**


- Read this entry for Set Theory.
- Follow links to explain specific concepts as you need to.

**Wikipedia (visited 29.08.06)**


- Type ‘Set Theory’ into the search box and press ENTER.
- Follow links to explain specific concepts as you need to.

**MacTutor History of Mathematics**


- Read for interest the history of Set Theory

**Composite Functions (visited 06.11.06)**

[http://www.bbc.co.uk/education/asguru/maths/13pure/02functions/06composite/index.shtml](http://www.bbc.co.uk/education/asguru/maths/13pure/02functions/06composite/index.shtml)

- Read through the first page
- Use the arrow buttons at the bottom of the page to move to the next page
- Page 2 is an interactive activity. Work through it carefully.
- Read page 3 for details on notation.
- Test your understanding on page 4.
**Wolfram MathWorld (visited 06.11.06)**

http://mathworld.wolfram.com/Composition.htm

- Read this entry for Composite Functions.
- Follow links to explain specific concepts as you need to.

**Wikipedia (visited 06.11.06)**

http://www.wikipedia.org/

- Type ‘Composite Functions’ into the search box and press ENTER.
- Follow links to explain specific concepts as you need to.

**Binary Color Device (visited 06.11.06)**


- This is a puzzle involving binary operations and group tables. Use the puzzle to develop your understanding.

**Wolfram MathWorld (visited 06.11.06)**

http://mathworld.wolfram.com/BinaryOperation.html

- Read this entry for Binary Operations.
- Follow links to explain specific concepts as you need to.

**Wolfram MathWorld (visited 06.11.06)**

http://mathworld.wolfram.com/Group.html

- Read this entry for Group Theory.
- Follow links to explain specific concepts as you need to.

**Wikipedia (visited 06.11.06)**

http://www.wikipedia.org/

- Type ‘Binary Operations’ into the search box and press ENTER.
- Follow links to explain specific concepts as you need to.

**Wikipedia (visited 06.11.06)**


- Read this entry for Group Theory.
- Follow links to explain specific concepts as you need to.
MacTutor History of Mathematics

http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Development_group_theory.html

Read for interest the history of Group Theory
XIV. Learning Activities

Module 1: Basic Mathematics

Unit 1, Activity 1: Sets and Functions

Specific Learning Objectives

By the end of this activity, the learner should be able to:

• Distinguish between a function and a general mapping
• Demonstrate relationship between sets and functions
• Give examples of sets of real numbers and some functions defined on such sets

Overview

The notions of a set and a function are the most fundamental concepts which together constitute the foundations of Mathematics. Indeed, different branches of Mathematics start with these two fundamental concepts.

In this activity, we are simply demonstrating how sets of objects are easily extracted from our surroundings. In particular, we are going to motivate the learner to be able to easily come up with examples of general mapping and functions defined on sets of real numbers.

We note that it is of great importance for the learner to be able to distinguish between a general mapping and a function diagrammatically. This will help the learner in grasping many properties about functions in higher courses.

Key Concepts

Function: This is a special type of mapping where an object is mapped to a unique image.

Mapping: This is simply a relationship between any two given sets

Proposition: This is a statement with truth value. Thus we can tell whether it is true or false

Set: This is a collection of objects or items with same properties
Readings

All of the readings for the module come from Open Source text books. This means that the authors have made them available for any reader to use them without charge. We have provided complete copies of these texts on the CD accompanying this course.


Internet Resources

Set Theory (visited 29.08.06)

http://www.mathresource.iitb.ac.in/project/indexproject.html

• Read through any of the sections by clicking on the slices of the pie.
• Especially work through the section called ‘functions’.
• Click the NEXT link at the bottom of the page to move forward.
• Click on double arrow buttons to see things move!

Wolfram MathWorld (visited 29.08.06)

http://mathworld.wolfram.com/SetTheory.html

• Read this entry for Set Theory.
• Follow links to explain specific concepts as you need to.

Wikipedia (visited 29.08.06)

http://www.wikipedia.org/

• Type ‘Set Theory’ into the search box and press ENTER.
• Follow links to explain specific concepts as you need to.

MacTutor History of Mathematics

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Beginnings_of_set_theory.html

• Read for interest the history of Set Theory
Introduction

a) Story of Maize Grinding Machine

Jane walks in a village to a nearby market carrying a basket of maize to be ground into flour. She puts the maize into a container in the grinding machine and starts rotating the handle. The maize is then ground into flour which comes out of the machine for her to take home.

Question
What relation can you make among the maize, the grinding machine and the flour?

b) Story of children born on the Christmas day in the year 2005

It was reported on the 25th of December 2005 in Pumwani Maternity Hospital which is in Nairobi the Capital City of Kenya that mothers who gave birth to single babies were a total of 52. This was the highest tally on that occasion. As it is always the case each baby was given a tag to identify him or her with the mother.

Questions
1. In the situation above given the mother how do we trace the baby?
2. Given the baby how do we trace the mother?
Activity

Note that we can now represent the story of the maize grinding machine diagrammatically as follows:

\[ A \rightarrow f \rightarrow B \]

A = Set of some content (in this case maize) to be put in the grinding machine.

\( f \) = The mapping or function representing the process in the grinding machine

B = Set of the product content (in this case flour) to be obtained
Example 1
In this example we define two sets and a relation between them as follows:
Let $A = \{2, 3, 4\}$
$B = \{2, 4, 6, 8\}$

$f$ is a relationship which says “is a factor of”
e.g. 3 is a factor of 6

In this case we have the following mapping:
Example 2

Think of a number of such situations and represent them with a mapping diagram as shown above.

In our second story of each mother giving birth to only one child can be represented in a mapping diagram as follows:

A = Set of babies
B = Set of mothers
\( f \) = Relation which says “baby to”
Remarks 3

i. Notice that in this mapping each object is mapped onto a unique image. In this case it is a function. We write \( f: A \rightarrow B \)

ii. Note also that in the mapping above even if we interchanged the roles of sets A and B we still have that each object has a unique image. Thus we have…

In this case we have

\[
\begin{align*}
B &= \text{Set of mothers} \\
A &= \text{Set of babies} \\
g &= \text{Relation which says “is mother of”}
\end{align*}
\]

In this case we say that the function \( f \) has an inverse \( g \). We normally denote this inverse \( g \) as \( f^{-1} \)

Thus for \( f: A \rightarrow B \) we have \( f^{-1}: B \rightarrow A \)
Example 4

Let \( A = \{1, 2, 3, 4, 5\} \)

\[ B = \{2, 3, 5, 7, 9, 11, 12\} \]

\[ f: x \mapsto 2x + 1 \]

Then we have the mapping as follows: \( f: x \mapsto 2x + 1 \)

For notation purposes in this mapping we can also write:

\[ f(1) = 3, f(2) = 5 \text{ etc} \]

In general, \( f(x) = 2x + 1 \)

The set \( A \) is called the domain of \( f \) and the set \( B \) is called the co-domain of \( f \).

The set \( \{3, 5, 7, 9, 11\} \) within \( B \) on which all elements of \( A \) are mapped is called the range of \( f \). Note that the inverse of \( f \) is given by \( f^{-1}(x) = \frac{x - 1}{2} \) and is also a function.
Exercise 5
Starting with the set
A = {2, 4, 7, 9, 11, 12} as the domain find the range for each of the following functions.

a) \( f(x) = 3x - 2 \)

b) \( g(x) = 2x^2 + 1 \)

c) \( h(x) = \frac{x}{1 - x} \)

Exercise 6
State the inverse of the following functions:

a) \( f(x) = 3 - \frac{2}{x} \)

b) \( g(x) = \frac{1}{1 - x} \)

c) \( h(x) = 3x^2 - 2 \)

Exercise 7
Using as many different sets of real numbers as domains give examples of the following:

a) A mapping which is not a function
b) A mapping which is a function

c) A function whose inverse is not a function
d) A function whose inverse is also a function

Demonstrate each example on a mapping diagram. If you are in a group each member should come up with an example of his or her own for each of the cases above.
Module 1: Basic Mathematics

Unit 1, Activity 2: Composite functions

Specific Objectives

By the end of this activity, the learner should be able to:

• Demonstrate a situation in which two consecutive instructions issued in two different orders may yield different results.
• Verify that two elementary functions operated (one after another) in two different orders may yield different composite functions.
• Draw and examine graphs of different classes of functions starting with linear, quadratic etc.

Overview

Composite functions are about combinations of different simple mappings in order to yield one function. The process of combining even two simple statements in real life situations in order to yield one compound statement is important. Indeed the order in which two consecutive instruction are issued must be seriously considered so that we do not end up with some embarrassing results.

In this activity we are set to verify that two elementary functions whose formulae are known if combined in a certain order will yield one composite formula and if order in which they are combined is reversed then this may yield a different formula.

We note here that it is equally important to be able to represent a composite function pictorially by drawing its graph and examine the shape. Indeed, the learner will be able to draw these graphs starting with linear functions quadratic and even trigonometric functions etc.

Key Concepts

Composite Function: This is a function obtained by combing two or more other simple functions in a given order.
Readings

All of the readings for the module come from Open Source text books. This means that the authors have made them available for any reader to use them without charge. We have provided complete copies of these texts on the CD accompanying this course.

1. Sets relations and functions by Ivo Duntsch and Gunther Gediga, Methodos publishers (UK) 2000. (File name on CD: Sets_Relations_Functions_Duntsch)

Internet Resources

Composite Functions (visited 06.11.06)
http://www.bbc.co.uk/education/asguru/maths/13pure/02functions/06composite/index.shtml

- Read through the first page
- Use the arrow buttons at the bottom of the page to move to the next page
- Page 2 is an interactive activity. Work through it carefully.
- Read page 3 for details on notation.
- Test your understanding on page 4.

Wolfram MathWorld (visited 06.11.06)
http://mathworld.wolfram.com/Composition.html

- Read this entry for Composite Functions.
- Follow links to explain specific concepts as you need to.

Wikipedia (visited 06.11.06)
http://www.wikipedia.org/

- Type ‘Composite Functions’ into the search box and press ENTER.
- Follow links to explain specific concepts as you need to.
Introduction

a) A Story of Nursery School Children

Two Children brother and sister called John and Jane go to a Nursery school called Little Friends.

One morning they woke up late and found themselves in a hurry to put on clothes and run to school, Jane first put on socks then shoes. But her brother John first put on shoes then socks. Jane looked at him and burst into laughter as she run to school to be followed by her brother.

Question
Why did Jane burst into laughter?

b) Story of a visit to a beer brewing factory

A science Club in a secondary school called Nabumali High School in Uganda, one Saturday made a trip to Jinja town to observe different stages of brewing beer called Nile Beer. It was noted that of special interest was the way some equipment used in the process would enter some chamber and emerge transformed. For example an empty bottle would enter a chamber and emerge transformed full of Nile Beer but without the bottle top. Then it would enter the next chamber and emerge with the bottle top on.

Question
Can you try to explain what happens in each chamber of the brewing factory?
Activity

We note that in our story of the Nursery school Children what is at stake is the order in which we should take instruction in real life situations. Jane laughed at her brother because she saw the socks on top of the shoes. In other words her brother had ended up with composite instruction or function which was untenable. We can also look at other such cases through the following example.

Example 1

I think of a number, square it then add 3 or I think of a number add 3 then square it. If we let the number to be $x$, then we will end up with two different results namely $x^2 + 3$ and $(x + 3)^2$ respectively.

Example 2

Can you now come up with a number of examples similar to the one above?

If we now consider our story on the brewing of Uganda Warangi we note that each Chamber has a specific instruction on the job to perform. This is why whatever item passes through the chamber must emerge transformed in some way. We can also look at an example where instructions are given in functional form with explicit formulae as shown below.

Example 3

Consider the composition of the functions.

$f : x \rightarrow 2x$ and $g : x \rightarrow x + 5$

Here if we are operating $f$ followed by $g$ then we double $x$ before we add 5. But if operate $g$ followed by $f$ then we add 5 to $x$ before we double the result.

For notation purposes

$(f \circ g)(x) = f(g(x))$ means $g$ then $f$. While $(g \circ f)(x) = g(f(x))$ means $f$ then $g$.

Thus we have:

```
g : x \rightarrow 2x
```

```
g : x \rightarrow x + 5
```

```
4 8 11
```
Representing the composite function \( g(f(x)) = 2x + 5 \)

While:

\[
\begin{array}{cc}
4 & 9 \\
\text{g: } x & \rightarrow x + 5 \\
\end{array}
\]

\[
\begin{array}{cc}
9 & 18 \\
\text{f: } x & \rightarrow 2x \\
\end{array}
\]

representing the composite function \( f(g(x)) = 2(x + 5) \)

**Exercise 4**

Given \( f: x \rightarrow 3x + 1 \quad g: x \rightarrow x - 2 \)

Determine the following functions:

(a) \( f \circ g \)

(b) \( g \circ f \)

(c) \( (f \circ g)^{-1} \)

(d) \( (g \circ f)^{-1} \)

Taking \( x = 3 \) draw a diagram for each of the composite functions above as is the case in example 3 above.

**Exercise 5**

Sketch the graph for each of the following function: assuming the domain for each one of them is the whole set \( \mathbb{R} \) of real numbers.

b) \( f(x) = 2x - 3 \)

c) \( g(x) = 4x^2 - 12x \)

d) \( h(x) = x^3 - 3x + 1 \)

e) \( k(x) = 2 \sin x \)
Module 1: Basic Mathematics

Unit 2: Binary Operations

Specific Learning Objectives

By the end of this activity, the learner should be able to:

• Give examples of binary operations on various operations
• Determine properties of commutativity or associativity on some binary operations.
• Determine some equivalence relations on some algebraic structures

Overview

The concept of a binary operation is essential in the sense that it leads to the creation of algebraic structures.

The well known binary operations like + (addition) and x (multiplication) do constitute the set $\mathbb{R}$ of real numbers as one of the most familiar algebraic structures. Indeed the properties of commutativity or associativity can easily be verified with respect to these operations on $\mathbb{R}$.

However, in this activity we define and deal with more general binary operations which are usually denoted by $\ast$.

For example for any pair of points x, and y, in a given set say G, x $\ast$ y could even mean pick the larger of the two points. It is clear here that x $\ast$ y = y $\ast$ x. Consequently we will exhibit examples of more general algebraic structures that arise from such binary operations.

Key concepts

Algebraic structure: This is the collection of a given set G together with a binary operation $\ast$ that satisfies a given set of axioms.

Binary operation: This is a mapping which assigns to each ordered pair of elements of a set G, exactly one element of G.
Readings

All of the readings for the module come from Open Source text books. This means that the authors have made them available for any reader to use them without charge. We have provided complete copies of these texts on the CD accompanying this course.

1. Sets of relations and functions by Ivo Duntsch and Gunther Gediga, Methodos Publishers, UK, 2000 pp 30-34 (File name on CD: Sets_Relations_Functions_Duntsch)

Internet Resources

Binary Color Device (visited 06.11.06)
- This is a puzzle involving binary operations and group tables. Use the puzzle to develop your understanding.

Wolfram MathWorld (visited 06.11.06)
http://mathworld.wolfram.com/BinaryOperation.html
- Read this entry for Binary Operations.
- Follow links to explain specific concepts as you need to.

Wikipedia (visited 06.11.06)
http://www.wikipedia.org/
- Type ‘Binary Operations’ into the search box and press ENTER.
- Follow links to explain specific concepts as you need to.
Introduction: The Story of the Reproductive System

In a real life situation among human beings, you will find that an individual gets into a relation with another individual of opposite sex. They then reproduce other individuals who constitute a family. We then have that families with a common relationship will constitute a clan and different clans will give rise to a tribe etc…

We note that even in ecology the same story can be told. For example we can start with an individual like an organism which is able to reproduce other organisms of the same species that will later constitute a population. If different populations stay together then they will constitute a community etc…

Question:

What is the mechanism that can bring together two individuals (human beings or organisms of ecology) in order to start reproduction?

Activity

We note that in the case of human beings in our story above we could say that it is marriage that brings together a man and woman to later constitute a family after reproduction. In mathematics the concept of marriage could be looked at as a binary operation between the two individuals. If we can reflect on our mapping diagram we have the following:

Where A = set of men wedding in a given time

B = set of women getting marriage at the same time

* = operation which says x weds y
Clearly \( x \ast y = y \ast x \)

In this case this particular binary operation is commutative. If we denote the relation implied by the binary operation \( \ast \) by \( R \) then we write \( xRy \) to mean \( x \) is related to \( y \) or \( yRx \) to mean \( y \) is related to \( x \).

If \( xRy \Rightarrow yRx \) then the relation is said to be symmetric.

**Question**

Can you try to define some relations on sets of your choice and check whether they are symmetric?

In general we note that if a binary operation \( \ast \) gives rise to a relation \( R \) then:

a) \( R \) is reflexive if \( xRx \)

b) \( R \) is symmetric if \( xRy \Rightarrow yRx \)

c) \( R \) is transitive if \( xRy \) and \( yRz \Rightarrow xRz \)

For all elements \( x, y, z \) in a given set

We also note that a relation \( R \) satisfying all the three properties of reflexive, symmetric and transitive above is said to be an equivalence relation.

**Example 1**

Let \( U \) be the set of all people in a community.

Which of the following is an equivalence relation among them?

i. is an uncle of

ii. is a brother of

We note that in part (i) if \( R \) is the relation “is an uncle of” then \( xRy \) does not imply \( yRx \).

Thus \( R \) is not symmetric in particular. Hence \( R \) is not an equivalence relation.

However in part (ii) if \( R \) is the relation “is a brother of” then \( xRx \) is valid.

Also \( xRy \Rightarrow yRx \) and finally \( xRy \) and \( yRz \Rightarrow xRz \).

Hence \( R \) is an equivalence relation.
Exercise 2

Which of the following is an equivalence relation on the set of all human beings?

i. is a friend of
ii. is a relative of

Exercise 3

a) Determine whether the binary operation \( * \) on the set \( \mathbb{R} \) of real numbers is commutative or associative in each of the following cases

i. \( x * y = y^2x \)
ii. \( x * y = xy + x \)

b) Define a relation \( \sim \) on the set of integers as follows

\( a \sim b \) if and only if \( a + b \) is even.
Determine whether \( \sim \) is an equivalence relation on \( \mathbb{R} \).

c) Give an example of an equivalence relation on the set \( \mathbb{R} \) of real numbers.
If you are working in a group each member of the group should give one such example.

d) Complete exercise 2.4.1 p 34 in Sets, Relations and Functions by Duntsch and Gediga (solutions on pp. 48 – 49)
Module 1: Basic Mathematics

Unit 3: Groups, Subgroups and Homomorphism

Specific Objectives

By the end of this activity, the learner should be able to:

- State axioms for both a group and a ring.
- Give examples of groups and subgroups.
- Give examples of rings and subrings.
- Give examples of homomorphisms between groups and isomorphisms between rings.
- Prove some results on properties of groups and rings.

Overview

Recall that in our Unit 2 activity 2 we looked at the case of an individual organism being able to reproduce and give rise to a population. Note that a population here refers to a group of individuals from the same species. In this activity we are going to demonstrate that a general algebraic structure can give rise to a specific one with well stated specific axioms.

We will also reflect on the notion of relations between sets using mappings, whereby we will define a mapping between any two given groups. It is at this stage that the concept of a homomorphism will come into play. The situation of looking at the properties of a mapping between two sets which are furnished with algebraic structure as the groups are can be of great interest and indeed it is the beginning of learning proper Abstract Algebra.
Key Concepts

**Abelian group:** This is a group $\langle G, \cdot \rangle$ in which $a \cdot b = b \cdot a$ for $a, b \in G$.

**Group:** This is a non-empty set say $G$ with a binary operation $\cdot$ such that:

(i) $a \cdot b \in G$ for all $a, b \in G$.

(ii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in G$.

(iii) There exists an element $e$ in $G$ such that $e \cdot a = a = a \cdot e$ for all $a \in G$ where $e$ is called identity.

(iv) For every $a \in G$ there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = e = a^{-1} \cdot a$.

Where $a^{-1}$ is called the inverse of $a$.

**Homomorphism:** This is a mapping $\phi$ from a group $G$ into another group $H$ such that for any pair $a, b \in G$. We have $\phi(xy) = \phi(x) \phi(y)$.

**Isomorphism:** This is a homomorphism which is also a bijection.

**Ring:** This is a non-empty $\mathbb{R}$ set say with two binary operations $+$ and $\cdot$ called addition and multiplication respectively such that:

(i) $\langle \mathbb{R}, + \rangle$ is an Abelian group.

(ii) $\langle \mathbb{R}, \cdot \rangle$ is a multiplicative semigroup.

**Semigroup:** This is a non-empty set $S$ with a binary operation $\cdot$ such that:

(i) $a \cdot b \in S$ for all $a, b \in S$.

(ii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in S$.

(iii) For all $a, b, c \in \mathbb{R}$ we have:

\[ a \cdot (b+c) = a \cdot b + a \cdot c \quad \text{and} \quad (a+b) \cdot c = a \cdot c + b \cdot c \]
**Subgroup:** This is a subset $H$ of a group $G$ such that $H$ is also a group with respect to the binary operation in $G$.

**Readings**

All of the readings for the module come from Open Source text books. This means that the authors have made them available for any reader to use them without charge. We have provided complete copies of these texts on the CD accompanying this course.

Abstract Algebra: The Basic Graduate Year, by Robert B. Ash (Folder on CD: Abstract_Algebra_Ash)

**Internet Resources**

**Wolfram MathWorld (visited 06.11.06)**


- Read this entry for Group Theory.
- Follow links to explain specific concepts as you need to.

**Wikipedia (visited 06.11.06)**


- Read this entry for Group Theory.
- Follow links to explain specific concepts as you need to.

**MacTutor History of Mathematics**

[http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Development_group_theory.html](http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Development_group_theory.html)

- Read for interest the history of Group Theory
Introduction: Story of a Cooperative Society

In 1990, one hundred workers of a certain Institution in Kenya decided to form a cooperative society called CHUNA in which they were contributing shares on a monthly basis. They set up rules for the running of the society which included terms for giving out loans. It was decided after they had run the society for sometime that the officials should pay regular visits to other well established cooperative societies in the country to see how they are run in comparison to there own.

It was noted after those visits to other societies that there was need to moderate some of there rules of running the society in order to create consistency.

Questions

1. Why did they set up rules after forming the cooperative society?
2. What significance could you attach to their visits to other cooperative societies?

Activity

In our story above we note that a cooperative society requires rules to create an operating structure. This is equivalent to having axioms that are satisfied by elements of a non-empty set as is the case with the group G.

Question

Can you now think of other situations where a group of people or objects could have sets of rules among them, which resemble the axioms of a group?

Example 1

Consider the set \( \mathbb{Z} \) of integers in order the operation of addition (+). We have that

(i) \( a + b \) in \( \mathbb{Z} \) for all \( a, b, \in \mathbb{Z} \)
(ii) \( a + (b + c) = (a + b) + c \) for all \( a, b, c \in \mathbb{Z} \)
(iii) there is \( 0 \in \mathbb{Z} \) such that \( a + 0 = a = 0 + a \) for all \( a \in \mathbb{Z} \)
(iv) for every \( a \in \mathbb{Z} \) there is \(-a\) such that \( a + -a = 0 = -a + a \)

Hence, \( \{ \mathbb{Z}, + \} \) is a group
Exercise 2

Verify that the set $R$ of real numbers is also a group under addition.

Note that if for any group $\{G, *\}$ we have that for any pair of points $x, y \in G$, $x * y = y * x$

Then $G$ is called Abelian group.

In this case the group $\{R^+, +\}$ is Abelian.

The second question coming out of our story of the cooperative society above is mainly for comparison purposes. This is to find out whether the structure set up by CHUNA compares well with those of other societies. Similarly the structures of groups are easily compared using mappings. Thus for any two given groups say $G$ and $H$ a mapping can be defined between them in order to compare their structures. In particular a homomorphism $\phi: G \rightarrow H$ is a mapping that preserves the structure. In other words $G$ and its image under $\phi$ (denoted by $\phi(G)$ in $H$ are the same group structurally. Note that if a homomorphism is an onto mapping then it is called an isomorphism.

Example 3

Let $G$ and $H$ be any two groups and $e^1$ be the identity of $H$. Then the mapping $\phi: G \rightarrow H$ given by

$$\phi(xy) = e^1$$

is a homomorphism.

Indeed for any pair $x, y \in G$,

$$\phi(xy) = e^1 = e^1 e^1 = \phi(x)\phi(y)$$

Exercise 4

Let $G$ be the group $\{R^+, \cdot\}$ of positive real numbers under multiplication and let $H$ be the additive group $\{R^+, +\}$ of real numbers. Show that the mapping:

$\phi: G \rightarrow H$ given by

$$\phi(xy) = \log_{10} x$$

is a homomorphism.
Remarks 5

1. Note that a subgroup \( H \) of \( G \) is a subject of \( G \) written \( H \subseteq G \) which is also a group with respect to the binary operation \( G \).

2. Note also that the identity element of a given subgroup say \( H \) of a group \( G \) is the same as the identity of the group \( G \).

3. Note therefore that all the considerations can also be pulled down on subgroups to find out what results can follow.

4. Note also that the theorem stated below is useful in determining subgroups.

Remarks 6

Let \( G \) be a group. A non-empty subset \( H \) of \( G \) is a subgroup of \( G \) if and only if \( a, b \in H \implies ab^{-1} \in H \)

Exercise 7

a. Let \( H \) and \( K \) be subgroups of a group \( G \). then show that \( H \cap K \) is also a subgroup of \( G \).

b. Let \( H \) be a subgroup of a group \( G \). Show that \( Ha = H \) if and only if \( a \in H \).

c. Let \( G \) and \( H \) be groups and \( \Phi : G \rightarrow H \) be a homomorphism. Show that \( \text{Ker} (\Phi) \) is a subgroup of \( H \).

Where \( \text{Ker}(\Phi) = \{ x \text{ in } G : \Phi(e) = e \} \)

\( \text{Im } \Phi = \{ \Phi(x) \text{ in } H : x \in G \} \)

Exercise 8

Read chapter 1 of Basic Algebra by Ash (from pp 1-18) and complete the exercise on p.19. Mark your own work from the answers chapter.
**XV. Synthesis of the Module**

*Synthesis Of The Basic Mathematics Module*

We note that having gone through this module you should now be fully equipped with the concepts involved in the following contents.

In Unit 1 the most basic concepts are those of a set and function, followed by logic in which you are introduced to the science of reasoning. A good grasp of the real number system is also necessary for easy definition of elementary functions. Permutations and combinations together with trigonometric functions complete the most significant topics in this unit. These concepts are given great exposition in this unit.

In Unit 2 you have been introduced to algebraic structures in which the concept of a binary operations played a pivotal role. The concept of equivalence relation is essential. This leads to partitioning of sets into equivalence classes that facilitates deeper studies on sets or collections of spaces.

Finally Unit 3 brings out study on two particular examples of algebraic structures namely groups and rings. It is important here to stress both their similarities and main differences. Indeed one similarity that the learner should have noticed is that in the structure of a group we can find a subgroup just as it is the case with a ring and a subring. However, the main difference between the two algebraic structures is that a group thrives only on one binary operation whereas a ring thrives on two binary operations. These facts are well exposed in the two learning activities in this unit. You should as a final consideration in this unit look at the definition of mappings from one group to another or from one ring into another ring. In particular a homomorphism which is known to preserve the structure of a given group is essential in this study.
XVI. Summative Evaluation

Module 1: Basic Mathematics

Unit 1: Summative Assessment Questions

Question 1

a. Write the negation of the following statement:
   If I receive a salary increase I will buy a plot.

b. Use truth tables to show that
   \[ A \lor (B \land C) \equiv (A \lor B) \land (A \lor C) \]

c. Determine the truth tables for the following propositions.
   i. \((A \Rightarrow B) \Rightarrow (A \lor B)\)
   ii. \((A \Rightarrow B) \lor (\sim A \land \sim B)\)

Question 2

a. Give the definition of a function. In the diagram below state with reasons whether the mapping represents a mapping or not.

b. Let \(A = \{x: -2 \leq x \leq 2\}\). Let \(f : A \rightarrow \mathbb{R}\) and \(g : A \rightarrow \mathbb{R}\) be defined by
\( f(x) = 3x + 4 \)
\( g(x) = (x - 1)^2 \)

Determine the range of each of the functions \( f \) and \( g \).

c. State the inverse of each of the following functions

i. \( f'(x) = 1 + \frac{1}{x} \)

ii. \( g(x) = \sqrt{3 + x^2} \)

**Question 3**

a. Let two functions \( f \) and \( g \) be defined on the whole set of real numbers by

\( f(x) = x - 1 \)
\( g(x) = 2x^2 \)

Find the composite functions (i) \( f \circ g \) and (ii) \( g \circ f \)

b. Given \( h(x) = x + 1 \) and \( g(x) = x^2 + 4 \) where each of these functions is defined on \( \mathbb{R} \) find

I. \( (h \circ g)^{-1} \)

II. The ranges of \( h \circ g \) and \( g \circ h \)

c. In part (b) above find the value of \( a \) such that \( (h \circ g)(a) = (g \circ h)(a) \)

**Question 4**

a. In how many ways can 6 boys be chosen from a class of 30 boys if the class captain is to be included?

b. A committee of six people is to be chosen from a group of 8 women and five men.

i. In how many ways can this be done?

ii. If one particular man must not be in the team how many of these teams will have more women than men?
c. A box contains 15 balls, 5 of which are red, 4 are green and 6 are blue. In how many ways can three balls be chosen if
   i. There is no restriction?
   ii. The balls must be of the same colour?
   iii. Only two balls are of the same colour?

**Question 5**

a) Simplify $\sin 4\theta - \sin 3\theta + \sin 2\theta$

b) Solve for $x$, $0 \leq x^o \leq 360^o$.
   i. $\sec^2 x - 5(\tan x - 1) = 0$
   ii. $2 \cos 2x + \sin^2 x = 2 \cos x$

c) Express $\sqrt{3}$ in the form $r \cos (\theta + \alpha )$
   Hence solve for $\theta$ in the range $0 \leq \theta \leq 180$ the equation $\sqrt{3} \cos \theta - \sin \theta = 0$
**Unit 1: Summative Assessment Solutions**

Q1. (a) If I do not receive a pay increase I will not buy a plot.

(b) 

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Q2

(a) A function is a mapping where each object is mapped onto a unique image. The diagram does not represent a function because there is an object mapping onto two different images.

(b) Given \( A = \{x : -2 \leq x \leq 2\} \)

\[
f(x) = 3x + 4
\]

\[
g(x) = (x-1)^2
\]

Range of \( f = \{y : -2 \leq y \leq 10\} \)

Range of \( g = \{y : 0 \leq y \leq 9\} \)

(c)

i. If \( f(x) = 1 + \frac{1}{x} \) then \( f^{-1}(x) = \frac{1}{x-1} \)

ii. If \( g(x) = \sqrt{3 + x^2} \) then \( g^{-1}(x) = \sqrt{x^2 - 3} \)

Q 3

(a) Given \( f(x) = x - 1 \)

\[
g(x) = 2x^2
\]

i. \( (f \circ g)(x) = f(g(x)) = f(2x^2) = 2x^2 = 2x^2 - 1 \)

ii. \( (g \circ f)(x) = g(f(x)) = g(x - 1) = 2(x - 1)^2 \)

(b) Given \( h(x) = x + 1 \)

\[
g(x) = x^2 + 4
\]

i. \( (h \circ g)(x) = h(g(x)) = h(x^2 + 4) = x^2 + 4 + 1 = x^2 + 5 \)

\( (h \circ g)^{-1}(x) = \sqrt{x - 5} \)
ii. Range of $h \circ g = \{y : y \geq 5\}$

Also $(g \circ h)(x) = g(h(x))$

$= g(x + 1)$

$= (x + 1)^2 + 4$

Range of $g \circ h = \{y : y \geq 4\}$

(a) $(h \circ g)(a) = a^2 + 5$

$(g \circ h)(a) = (a + 1)^2 + 4$

Therefore $(a + 1)^2 + 4 = a^2 + 5$

$2a + 5 = 5$

$2a = 0$

$a = 0$

Q 4

(a) If the captain is to be included then we are selecting 5 boys from a class of

25. Hence we have $^{25}C_5 = \frac{25!}{5!20!}$

$b = \frac{13!}{6!7!}$

(b) (i) This can be done in $= 13 \times 12 \times 11$

$= 1716$ ways

(i) We exclude the man who should not be in the committee leaving us with four men and eight women to choose from. Given that women have to be more than the men we have the following three options:

Either to select 4 women and 2 men giving us $^8C_4 \cdot ^4C_2$ ways

Or to select 5 women and 1 man giving us $^8C_5 \cdot ^4C_1$
Or to select 6 women and no man giving us \( ^8 C_6 \) ways

In total we have

\[
^8 C_4 x^4 C_2 + ^8 C_5 x^4 C_1 + ^8 C_6
\]

\[
= 420 + 224 + 28 = 672 \text{ ways}
\]

(c)

i. \( ^{15} C_5 \) ways

ii. \( ^5 C_1 x^6 C_5 x^4 C_1 = 120 \text{ ways} \)

iii. We have either 2G and 1R or 2G and 1B and 1G or 2B and 1R and 1G or 2R and 1G or 2R and 1B.

Q5

(a) We first note that \( \sin 4\theta + \sin 2\theta = 2 \sin \theta \cos \theta \)

Therefore \( \sin 4\theta - \sin 3\theta + \sin \theta = 2 \sin 3\theta \cos \theta - \sin 3\theta \)

\[
= \sin 3\theta (2 \cos \theta - 1)
\]

(b) (i) \( \sec^2 x - 5 (\tan x - 1) = 0 \)

\[
1 + \tan^2 x - 5 (\tan x - 1) = 0
\]

\[
\tan^2 x - 5 \tan x + 6 = 0
\]

Let \( y = \tan x \)

\[
y^2 - 5y + 6 = 0
\]

\[
(y - 3)(y - 2) = 0
\]

Therefore \( y = 3 \) or \( y = 2 \)

\[
\tan x = 3 \quad \text{or} \quad \tan x = 2
\]

\[
x = \tan^{-1} 3 \quad \text{or} \quad x = \tan^{-1} 2
\]
(ii)

(c) \(2(\cos^2 x - 1) + 1 - \cos^2 x = 2\cos x\)

\[\sqrt{3}\cos\theta - \sin\theta = r\cos(\theta + \alpha) = r\cos\theta \sin\alpha - r\sin\theta \cos\alpha\]

\[r\sin\alpha = \sqrt{3}\ and\ r\cos\alpha = 1\]

\[\therefore r = \pm 2\ and\ \tan\alpha = \frac{1}{\sqrt{3}} \therefore \alpha = 30^\circ\]

\[\pm 2\cos(\theta + 30^\circ) = 0\]
\[\Rightarrow \cos(\theta + 30^\circ) = 0\]
\[\therefore \theta + 30^\circ = 180^\circ\]
\[\therefore \theta = 150^\circ\ for\ 0 \leq \theta \leq 180^\circ\]
Unit 2: Summative Assessment Questions

1. Determine whether the binary operation $\ast$ on the set $\mathbb{R}$ of real numbers is commutative or associative in each of the following cases

   (a) $x \ast y = x^2 y$
   (b) $x \ast y = xy + y$

2. (a) Let $S$ be a non empty set with an associative binary operation $\ast$ on it. For $x, y, z \in S$ suppose that $x$ commutes with $y$ and $z$. Show that $x$ also commutes with $y \ast z$.

   (b) Prove that if $a, b \in \mathbb{Z}$ such that $a \parallel b$ and $a \parallel c$ then $a \parallel mb + nc$ for $m, n \in \mathbb{Z}$. Where $a \parallel b$ means $a$ divides $b$.

3. Give the definition of an equivalence relation. Define a relation $\sim$ on the set $\mathbb{Z}$ of integers as follows:

   $a \sim b$ if and only if $a + b$ is even. Show that $\sim$ is an equivalence relation on $\mathbb{Z}$.

4. (a) The relation congruence modulo $n$ on the set $\mathbb{Z}$ of integers is defined as follows:

   For any pair $x, y \in \mathbb{Z}$ $x$ is said to be congruent to $y$ modulo $n$ written $x \equiv y \pmod{n}$ if $n$ divides $x - y$.

   Show that this is an equivalence relation.

   (b) Show that the relation $\sim$ defined on $\mathbb{N} \times \mathbb{N}$ by $(a, b) \sim (c, d)$ iff $a + d = b + c$ is an equivalence relation. Where $\mathbb{N}$ is the set of natural numbers.
Unit 2: Summative Assessment Answers

1. (a) Let \( x, y, z \in \mathbb{R} \) Then \( x \ast (y \ast z) = x^2 \ast (y \ast z) = \left( x^2 \right)^2 y^2 z = x^4 y^2 z \).

Also \((x \ast y) \ast z = x^2 y \ast z = \left( x^2 y \right)^2 z = x^4 y^2 z\)

Thus \( x \ast (y \ast z) = (x \ast y) \ast z \)

Hence \( \ast \) is associative.

We also have that:
\( x \ast y = x^2 y \) and \( y \ast x = y^2 x \)

Thus \( x \ast y \neq y \ast x \)

Hence \( \ast \) is not commutative.

(b) Let \( x, y, z \in \mathbb{R} \) Then

\( x \ast (y \ast z) = x \ast (yz + z) = x(yz + z) + yz + z = xyz + zx + yz + z \)

Also

\((x \ast y) \ast z = (xy + y) \ast z\\ \quad = (xy + y)z + z\\ \quad = xyz + yz + z\\ \therefore x \ast (y \ast z) \neq (x \ast y) \ast z\)

Hence \( \ast \) is not associative.

We also have that:
\( x \ast y = xy + y \) and \( y \ast x = yx + x \)

\( \therefore x \ast y \neq y \ast x \)

Hence \( \ast \) is not commutative.
2. (a) Given that:
\[ x * y = y * x \quad \text{and} \quad x * z = z * x \]
We have by associativity of * that

\[ x * (y * z) = (x * y) * z \]
\[ = (y * x) * z \]
\[ = y * (x * z) \]
\[ = y * (z * x) \]
\[ = (y * z) * x \]

Hence \( x \) also commutes with \( y * z \).

(b) Now \( a/b \Rightarrow b = k a \quad \text{and} \quad a/c \Rightarrow b = h a \quad \text{for some} \ k, h \in \mathbb{Z} \)

\[ mb + nc = mk a + nh a = a( mk + nh ) \]
\[ \Rightarrow a/m b + nc \]

3. An equivalence relation is one which is reflexive symmetric and transitive.

Given the relation:
\[ a \sim b \iff a + b \text{ is even.} \]
we have:

(i) \( a + a = 2a \Rightarrow a \sim a \quad \text{reflexive} \)

(ii) \( a \sim b \Rightarrow a + b = 2k \Rightarrow b + a = 2k \)
\[ \Rightarrow b \sim a \quad \text{symmetric} \]

(iii) \( a \sim b \text{ and } a + c = 2m \text{ and } b + c = 2n \)
\[ \Rightarrow a + c = 2m - b + 2n - b \]
\[ = 2m + 2n - 2b. \]
\[ = 2(m + n - p) \Rightarrow a \sim c. \]

transitive

Hence \( \sim \) is an equivalence relation.
4. (a) For any pair \( x, y \in \mathbb{Z} \) we have:

(i) \( n \) divides reflexive \( x - x = 0 \Rightarrow x \equiv x \pmod{n} \)

(ii) Let \( n \) divide \( x - y \). Then also \( n \) divides \( y - x \) which is symmetric property.

(iii) Let \( n \) divide \( x - y \) and also \( y - z \). Then also \( n \) divides \( (x - y) + (y - z) = x - z \) which is transitive property. Hence congruence modulo in an equivalence relation.

(b) We note that:

(i) \( (a \sim b) \sim (a, b) \) since \( a + b = b + a \) which is reflexive property.

(ii) \( (a, b) \sim (c, d) \Rightarrow a + d = b + c \)

\[ \Rightarrow d + a = c + b \]

\[ \Rightarrow (c, d) \sim (a, b) \]

which is symmetric property.

(iii) Now suppose \( (a, b) \sim (c, d) \) and \( (c, d) \sim (e, f) \)

Then we have that:

\[ a + d = b + c \text{ and } c + f = d + e \]

\[ \Rightarrow a + d + c + f = b + c + d + e \]

\[ \Rightarrow a + f = b + e \Rightarrow (a, b) \sim (e, f) \]

Hence \( \sim \) is an equivalence relation.
Unit 3: Summative Assessment Questions

1. (a) Let \( G \) be a group such that \( a^2 = e \) for all \( a \in G \) Show that \( G \) is Abelian.

(b) Let \( G \) be a group such that \( (ab)^2 = a^2 b^2 \) for all \( a, b \in G \). Show that \( G \) is Abelian.

2. Given the matrices

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

verify that they form a multiplicative group.

2. (a) Show that if \( G \) is a group of even order then there are exactly an odd number of elements of order 2.

(b) If \( a \) and \( b \) are any two elements of a group \( G \), show that the order of \( ab \) is the same as the order of \( ba \).

3. (a) If \( \phi \) is an isomorphism of a group \( G \) into a group \( H \), prove that

\[
\phi \left( a^0 \right) = e \iff a^n = e .
\]

(b) Prove that the multiplicative group of the \( n^{th} \) root of unit is a cyclic group of order \( n \).

4. (a) Let \( \mathbb{C} \) denote the complex number field.

Define \( A = \{ a + ib : a, b \in \mathbb{C} \} \) 

where \( i = \sqrt{-1} \).

Show that \( A \) is a subring of \( \mathbb{C} \).

(b) If \( a \) and \( b \) are nilpotent elements of a commutative ring, show that \( a + b \) is also nilpotent.

Give an example to show that this may fail if the ring is not commutative.
Unit 3: Summative Assessment Solutions

1 (a) We note that for all \( a, b, \in G \), we have that:

\[ a, b, \in G \Rightarrow a, b, \in G \land b^2 = e \text{ since } a^2 = e \text{ since } a^2 = e \text{ for all } a \in G. \]

\[ (ab)^2 = e. \]

Thus

\[ ab = a \cdot eb \]
\[ = a(ab)^2 b \]
\[ = a(ab \cdot ab)b \]
\[ = a^2b \cdot ab^2 \]
\[ = eb \cdot a \cdot e \]
\[ = ba \]

Hence G is Abelian.

(b) Given that:

\[ (ab)^2 = a^2b^2, \]
\[ ab \cdot ab = aa \cdot bb. \]

Apply cancellation law to give \( ba = ab \).

Hence G is Abelian.

2. Letting \( e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

\[ a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
\[ b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]
\[ c = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \]
We have:

\[ a^2 = b^2 = c^2 = e \]
\[ ab = c = ba \]
\[ bc = a = cb \]
\[ ca = b = ac \]

Thus \( G = \{ e, a, b, c \} \) is closed under multiplication. Now \( e \) is the identity on \( G \) and each member of \( G \) is an inverse of itself.

We also note that matrix multiplication is associative and therefore \( G \) is a multiplicative group.

3. (a) Suppose \( x \in G \) is not an element of order 2. Then \( x^2 \neq e \Rightarrow x \neq x^{-1} \)

It now follows that there is an even number of elements \( x \) such that \( x^2 \neq e \).

Thus there is an even number of elements \( x \) in \( G \) such that \( x^2 = e \)

Now since \( e \) also satisfies \( e^2 = e \) we have that there is an odd number of elements \( x \) in \( G \) which are of order 2.

(b) Suppose for the element \( ab \) in \( G \) order \( ab = m \). i.e. \( O(ab) = m \).

Then we have:

\[
(ba)^m = a^{-1} a (ba)^m = a^{-1} a ba ba \ldots . ba
\]

\[ m \text{ times} \]

\[ = a^{-1} (ab ab \ldots . ab)a \]

\[ m \text{ times} \]

\[ = a^{-1} (ab)^m = e \]

This implies \( O(ba) \) divides \( m = O(ab) \).

Similarly \( O(ab) \) divides \( O(ba) \Rightarrow O(ab) = O(ba) \).
4. (a) Suppose \( \psi(a)^n = e \). Then

\[
\psi(a)\psi(a)\psi(a)\ldots\psi(a) = \psi(a^n) = \psi(e).
\]

Thus \( a^n = e \) since \( \Psi \) is an isomorphism. Conversely suppose \( a^n = e \).

Then we have that:

\[
e = \psi(e) = \psi(a^n) = \psi(a)\psi(a^{n-1}) = \psi(a)\psi(a^{n-2})\ldots\psi(a)\]

Hence the proof.

(b) We first note that the roots of the equation

\[ z^n = 1 \]

where \( z \) is a complex variable are found by use of DeMoivre’s theorem to be

\[
\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}
\]

for \( k = 0, 1, 2, \ldots, n - 1 \)

these roots form a cyclic group order \( n \) generated by

\[
w = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.
\]

5. (a) Let \( a + bi \in A \) and \( c + di \in A \). Then

\[
(a + bi) - (c + di) = (a - c) + (b - d)i \in A.
\]

Also

\[
(a + bi)(c + di) = (ac - bd) + (cb + bc)i \in A.
\]

Hence \( A \) is a subring of the ring \( \mathbb{C} \) of complex numbers.
(b) Let \( a^m = 0 \) and \( b^n = 0 \). Also let \( k = \max \{m, n\} \). Then

\[
(a + b)^{2k} = \sum_{r=0}^{2k} \binom{2k}{r} a^{2k-r} b^r = 0.
\]

Hence \( a + b \) is also nilpotent.

**Example**

Consider a 2 x 2 matrix ring over a field \( F \) if

\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Clearly this ring is not commutative. We also have \( a^2 = b^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \). But \( a + b \) is not nilpotent.
XVII. References


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XVIII. Main Author of the Module

The Author

The author of this module on Basic Mathematics was born in 1953 and went through the full formal education in Kenya. In particular he went to University of Nairobi from 1974 where he obtained Bachelor of Science (B.Sc) degree in 1977. A Master of Science (M.Sc.) degree in Pure Mathematics in 1979 and a Doctor of Philosophy (P.HD) degree in 1983. He specialized in the branch of Analysis and has been teaching at the University of Nairobi since 1980 where he rose through the ranks up to an Associate Professor of Pure Mathematics.

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