Foreword

The African Virtual University (AVU) is proud to participate in increasing access to education in African countries through the production of quality learning materials. We are also proud to contribute to global knowledge as our Open Educational Resources are mostly accessed from outside the African continent.

This module was developed as part of a diploma and degree program in Applied Computer Science, in collaboration with 18 African partner institutions from 16 countries. A total of 156 modules were developed or translated to ensure availability in English, French and Portuguese. These modules have also been made available as open education resources (OER) on oer.avu.org.

On behalf of the African Virtual University and our patron, our partner institutions, the African Development Bank, I invite you to use this module in your institution, for your own education, to share it as widely as possible and to participate actively in the AVU communities of practice of your interest. We are committed to be on the frontline of developing and sharing Open Educational Resources.

The African Virtual University (AVU) is a Pan African Intergovernmental Organization established by charter with the mandate of significantly increasing access to quality higher education and training through the innovative use of information communication technologies. A Charter, establishing the AVU as an Intergovernmental Organization, has been signed so far by nineteen (19) African Governments - Kenya, Senegal, Mauritania, Mali, Cote d’Ivoire, Tanzania, Mozambique, Democratic Republic of Congo, Benin, Ghana, Republic of Guinea, Burkina Faso, Niger, South Sudan, Sudan, The Gambia, Guinea-Bissau, Ethiopia and Cape Verde.

The following institutions participated in the Applied Computer Science Program: (1) Université d’Abomey Calavi in Benin; (2) Université de Ouagadougou in Burkina Faso; (3) Université Lumière de Bujumbura in Burundi; (4) Université de Douala in Cameroon; (5) Université de Nouakchott in Mauritania; (6) Université Gaston Berger in Senegal; (7) Université des Sciences, des Techniques et Technologies de Bamako in Mali (8) Ghana Institute of Management and Public Administration; (9) Kwame Nkrumah University of Science and Technology in Ghana; (10) Kenyatta University in Kenya; (11) Egerton University in Kenya; (12) Addis Ababa University in Ethiopia (13) University of Rwanda; (14) University of Dar es Salaam in Tanzania; (15) Universite Abdou Moumouni de Niamey in Niger; (16) Université Cheikh Anta Diop in Senegal; (17) Universidade Pedagógica in Mozambique; and (18) The University of the Gambia in The Gambia.

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Course Overview

Welcome to applied calculus for computing

Calculation intended to be a general method of solving quantifiable problems.

in the application of the calculation method, or as it is known the "infinitesimal" method, a problem is "divided into infinitesimal parts" (differentiation), analyzed in its relations with the neighboring parts and then "added" (integration) until the solution method:...

the two parts of this the analysis and synthesis form a model for more sophisticated methods based on calculation, used in applied science

concepts you learn in calculus allow statistical, physicists and engineers create mathematical models of real situations and real problems and simulate their resolutions under different operating conditions.

the calculation was invented by Leibnitz (1684), but the method of application of the results will have an impact from the publication of the book Newton "the Mathematical Principles of Natural Philosophy" in 1687.

See:

- [http://en.wikipedia.org/wiki/Leibniz%E2%80%93Newton_calculus_controversy](http://en.wikipedia.org/wiki/Leibniz%E2%80%93Newton_calculus_controversy)
- [https://archive.org/details/newtonspmathema00newtrich](https://archive.org/details/newtonspmathema00newtrich)

This course intended as an introduction to this method, an entry for the first part of research and scientific research Differentiation:

are treated the following subjects:

- Basic Math: Numbers and simple functions, trigonometry, complex functions and analytic geometry;
- Functions derivative including trigonometric, exponential and logarithmic;
- Differentiation methods,
- Differentiation rules: product rule, quotient rule, chain rule; Higher order derivatives at first;
- Applications of differentiation: Maximum and minimum method to solve equations,
- Integral calculus: Primitivation as inverse operator of differentiation, differential, integral forms of Riemann, definite integrals, standard forms,
- Integration techniques: Integration by parts, substitution, partial fractions, numerical integration elements,
- Integration of applications: average value of a function, length calculation, area and volume
Course Overview

Prerequisites
Basic algebra Notions (operations on numerical sets N, Z and Q) . trigonometry

Materials
The materials needed for this course include:

- CALCULATING WITH ANALYTIC GEOMETRY, 3rd Edition, Volume 1, Louis Leithold, Publisher Harbra Ltda, São Paulo, 1994
- CALCULUS WITH ANALYTIC GEOMETRY, 2nd Ed, Simmons, G, McGraw-Hill, 1996;
- http://www.youtube.com/channel/UC4atV0sjMDVUW1v2ebQkCmQ http://www.wikipedia.org/
- Software geogebra (http://www.geogebra.org/) wxMaxima(http://sourceforge.net/projects/wxMaxima/) (or other CAS available under CC)

Course Objectives
By the end of this module, the student should be able to:

- determine the general behavior of a function by its graph, studying the sign function between the zeros, field edges and / or points of discontinuity;
- determine the points where a function reaches extreme values;
- derive elementary functions and derivatives of implicit functions;
- perform on primitives of elementary functions, integral of a differential form using replacement methods and integration by parts;
- operate on areas of plane figures bounded by function graph
- apply the calculation method in one-dimensional problems

Units
Drive 0: Diagnostic
This unit presents a review and makes the diagnosis of the basics of algebra (operations on numerical sets N, Z, Q and R) and trigonometry, while familiarizes the student with the specialized language of mathematics

Unit 1: functions and limits of functions
This unit introduces the basic elements of R topology, presents the elementary real functions and develops methods for solving equations and inequalities in R;
Also introduces sequences in R (N functions to R) studies the convergence of sequences of R, and finally studies limits of real functions either in real points or at infinity
**Unit 2: differential Calculation**

This unit has continuity functions, derived functions, the geometric INTERPRETATION derived, the derivative exterior differential forms of functions and the basic elements of differential equations of first order.

**Unit 3: integral calculus in**

This unit features the primitive of elementary functions, the integral of a differential form, the fundamental theorem of calculus, techniques integration, Riemann integral, the geometric interpretation of the Riemann integral and the application of the calculation method in the calculation of areas and volumes of figures and elementary geometric solids;

**Evaluation**

In each unit are included formative assessment tools to check the progress of a student's. At the end of each module are presented summative assessment tools, such as testing and final works, which comprise the knowledge and skills studied in the module.

The implementation of the instruments of summative it is at the discretion of the institution offering the course. The suggested evaluation strategy is as follows:

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### Course Overview

#### Timing

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<td>Unit 2: Differential Calculation</td>
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<td>Lectures + Practice + Aval. training</td>
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<tr>
<td>APPRAISAL</td>
<td>Summative assessment</td>
<td>12 hours</td>
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#### Readings and other resources

Readings and other features of this course are:

**Unit 0**

Readings and other required resources:

- Precalculus, Cynthia Young, John Wiley & Sons Inc., 2010;
- ALGEBRA & Trigonometry, 3 Ed, Cynthia Young, John Wiley & Sons Inc., 2013;
- [https://pt.khanacademy.org/math/](https://pt.khanacademy.org/math/)
- Readings and other optional features:
  - [http://www.youtube.com/channel/UC4atV0sjMDVUW1v2ebQkCmQ](http://www.youtube.com/channel/UC4atV0sjMDVUW1v2ebQkCmQ)

**Unit 1**

Readings and other required resources:

- CALCULATING WITH ANALYTIC GEOMETRY, 3rd Edition, Volume 1, Louis Leithold, Publisher Harbra Ltda, São Paulo, 1994
- CALCULUS WITH ANALYTIC GEOMETRY, 2nd Ed, Smmons, G, McGraw-Hill, 1996;
readings and other resources opcionais:

- https://pt.khanacademy.org/math/
- http://www.youtube.com/channel/UC4atV0sjMDVvW1v2ebQkCmQ

Unit 2

Readings and other required resources:

- CALCULATING WITH ANALYTIC GEOMETRY, 3rd Edition, Volume 1, Louis Leithold, Publisher Harbra Ltda, São Paulo, 1994
- CALCULUS WITH ANALYTIC GEOMETRY, 2nd Ed, Summons, G, McGraw-Hill, 1996;

readings and other resources optional:

- https://pt.khanacademy.org/math/
- http://www.youtube.com/channel/UC4atV0sjMDVvW1v2ebQkCmQ

Unit 3

Readings and other required resources:

- CALCULATING WITH ANALYTIC GEOMETRY, 3rd Edition, Volume 1, Louis Leithold, Publisher Harbra Ltda, São Paulo, 1994
- CALCULUS WITH ANALYTIC GEOMETRY, 2nd Ed, Summons, G, McGraw-Hill, 1996;

readings and other resources opcionais:

- http://www.youtube.com/channel/UC4atV0sjMDVvW1v2ebQkCmQ
Unit 0. Diagnostic

Introduction to Unit

In this unit is carried out a review and an assessment of the knowledge of the basic prerequisites of the calculation, including algebra (operations on numerical sets N, Z, Q and R) and trigonometry.

Unit Objectives

By the end of this unit, you should be able to:

- execute algebraic operations in numerical sets, in particular powers, rootedness and logarithms R;
- determine the cases in which certain algebraic operations are not set;
- determine the main values of trigonometric functions using the unit circle

key Terms

Figures: Result counts or measurements

Natural numbers: result of counts:

Mathematical induction: N method to compare with other sets

Binary operation: result of cj elements of operation. ... Is within cj

Operands: Elements that go in the transaction

Sum: successively repeating the counting

Multiplication unit: successively summing the identical portions

Potentiation: successive multiplication identical terms:

Commutative: the operands order may be changed

Associative: the combination of operands can be exchanged

Power: second operand of the potentiation:

Polynomialx variable in basis:
Learning Activities

Activity 1 - Algebra prerequisites Review

Introduction

The first part of this activity is the numbers, operations on numerical sets, inverse operations, not defined in A.

second part is the unit circle and defining as radian measure an angle, trigonométrias functions trigonometric identities and-.

business details

part I Algebra R

numbers result scores or measurements. Any of these activities is a comparison of a quantity (to be counted or measured) with a drive (the default count or measurement).

In the set of natural numbers, \( N = \{1,2,3,\ldots\} \) and \( N = \{1,2,3,\ldots\} \) sets- if the operation amount as the “successive repeat unit,” the operation multiplication as the “sum of successive identical portions,” and the enhancement operation as such “successive identical multiplication terms.”
The inverse operations are not always being necessary extend the set

\[ \mathbb{N} = \{1, 2, 3, \ldots\}, \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \]

for this to this

\[ \mathbb{Q} = \{\ldots, -\frac{1}{2}, \ldots, 0, \frac{1}{2}, \ldots\} \]

and finally \( \mathbb{Q} \) to \( \mathbb{R} \) to ensure existência results of inverse operations.

still further there are exceptions where the reverse operations are not defined in \( \mathbb{R} \). operations with \( \mathbb{N} \) and \( \mathbb{Z} \) are well known and will not be reviewed here.

Any element \( \mathbb{Q} \) can be represented by fractions \( \frac{p}{q} \) or \( p / q \), \( p, q \mathbb{Z} \), \( q \neq 0 \) where \( q \) \( p \) is called numerator denominator of the fraction defined:

are equal fractions (in the sense of equivalence classes): \( \frac{p}{q} = \frac{r}{s} \) \( ps = qr \), \( q \neq 0 \), \( s \neq 0 \)

inverse of fractions \( \frac{p}{q} \) \( ^{-1} = \frac{q}{p} \), \( q \neq 0 \), \( R \neq 0 \)

to add fractions become fractions in other equivalents, common denominator \( \frac{p}{q} + \frac{r}{s} = \left(\frac{ps + qr}{qs}\right) \), \( q \neq 0 \), \( s \neq 0 \)

Any fraction can be reduced to form \( \frac{p}{q} \), \( q \neq 0 \) where \( p, q \) are coprime (no common divisors except 1).

compatibility rules of \( \mathbb{Q} \) element with the potentiation \( n \mathbb{N} \), \( q \neq 0 \), \( w \neq 0 \):

\[
\left(\frac{p}{q}\right)^n = \frac{(p^n)}{(q^n)}
\]

\[
\left(\frac{p}{q}\right)^{-n} = \left(\frac{q}{p}\right)^{-n} = \frac{(q^n)}{(p^n)}
\]

The root extraction is defined as “potentiation the reverse

\[
a^{\frac{1}{n}} = x \rightarrow x = a^{\frac{n}{n}}
\]

to \( a \geq 0 \) if \( n \) is par

for \( a \in \mathbb{Q} \), if \( n \) is odd

Uses it is also the notation: \( a^{\frac{1}{n}} = [n] \sqrt[n]{a} \) and \( [2] \sqrt{a} = \sqrt{a} \)

with respect the definition of the conditions, compatibility of the rules of the elements, with the root extraction \( b \in \mathbb{Q} \) to \( n \in \mathbb{N} \), \( a > 0 \), \( b > 0 \), são:

- \([n] \sqrt[n]{a} [n] \sqrt[n]{b} = [n] \sqrt[n]{ab} = (ab)^{\frac{1}{n}}\)
- \([n] \sqrt[n]{a} [m] \sqrt[m]{a} = ([nm] \sqrt[n]{a})^{\frac{m}{m+n}}\)
- \([n] \sqrt[n]{[m] \sqrt[m]{a}} = [nm] \sqrt[n]{a} = a^{\frac{n}{mn}}\)
- \([n] \sqrt[n]{a} [m] \sqrt[m]{a} = a^{\frac{m}{m}}\)
- \((1)/(n) \sqrt[n]{a}) = (([n] \sqrt[n]{a})^{\frac{1}{n}})/a \) (rationalization of the fraction)
However, the root extraction is not binary operation in \( Q \), for example, \( \sqrt{2} \notin Q \).

The logarithm the base \( b \) is defined by: \( \log_\{a\} b = \{x\} x = a^x \) for all \( a > 0 \) and \( b > 0 \).

With respect the setting conditions, the compatibility rules for elements with \( Q \) logarithm for \( a > 0 \), \( q > 0 \), \( p > 0 \), são:

\[
\log_\{a\} pq = \log_\{a\} p + \log_\{a\} q
\]

\[
\log_\{a\} p = n \log_\{a\} p
\]

\[
\log_\{a\} (p/q) = \log_\{a\} p - \log_\{a\} q
\]

\[
\log_\{a\} p \log_\{b\} a = \log_\{b\} p
\]

Logarithm also is not a binary operation in \( Q \), for example, \( \log_2 3 \notin Q \) multiplication.

In the real body \( R \) can be carried out the following operations, ensuring that the result there is a real number and is unique:

addition,

subtraction,

division (except by zero),

real power of any positive basis or full power over any real basis (except power zero base of zero),

entire root extraction, for any real number is the index of an odd but only for nonnegative numbers if the index is even;

positive logarithm base (other than 1) for any positive real number operations:

is not set in the real body

division by zero,

power zero based zero;

root extraction couple of negative numbers;

basic logarithms of 1 or positive basis for nonpositive real numbers, and

negative base logarithms \( x \).

Part II - trig

Consider a displacement on the unit circle (radius 1) centered at a point (origin), from a reference direction (abscissa axis .)

The measurement of the angular displacement of the unit circumference is called angle, and the angle corresponding to a displacement unit on the unit circle is called radian:
another angle measurement unit is the degree: 2 radian = 360 degrees.

several intermediate positions can be identified on the unit circumference (note the symmetry of the circumference and the positions of points in the four quadrants):

the cos function and sin are defined as measures of abscissa and ordinate respectively of a point on the unit circle, moved from an angle relative to the axis x unit:

the most important trigonometric relationship comes this definition and the Pythagorean theorem applied to triangle cos sides and sin and hypotenuse equal to the \( \cos^2x + \sin^2x = 1 \) the second formula enough useful is \( \cos^2x - \sin^2x = \cos 2x \) the cos functions and sin are periodic with period equal to the angle after a full turn (2 rad). The graphs of sine and cosine functions, taking x as value of angle:
For the symmetry of the unit circle, it is only necessary to know the sin values and cos in the first quadrant:

\[
\cos (\pi - \theta) = - \cos \theta \\
\sin (\pi - \theta) = \sin \theta \\
\cos (-\theta) = \cos \theta \quad \text{(cosine is an even function)} \\
\sin (-\theta) = - \sin \theta \quad \text{(sine is an odd function)} \\
\cos ((\pi / 2) - \theta) = \sin \theta \quad \text{(sine and cosine are complementary)} \\
\sin ((\pi / 2) - \theta) = \cos \theta
\]

Some properties of elementary trigonometric functions:

zeros of \( \cos x \) are the values \( x = (\pi / 2) + n\pi \) to \( n\mathbb{Z} \)
the sin of zeros \( x \) are the values \( x = n\pi \) to \( n\mathbb{Z} \) 
both are periodic with the same period \( 2\pi \): \( \cos (x + 2\pi) = \cos x \) and \( \sin (x + 2\pi) = \sin x \) 
both are translations of each other: \( \cos (x + (\pi / 2)) = \sin x \) \( \sin (x - (\pi / 2)) = \cos x \) 
the tangent function is defined by the ratio of sine and cosine: \( \tan x = (\sin x) / (\cos x) \)
Note that the function is not defined in the cosine of zeros.

**Conclusion**

we reviewed the concepts of algebra numbers (measures on the real line) and trigonometry angles (measures on the unit circle).

we reviewed the main operations, its operations and inverse major exceptions / prohibitions.

**Rating**

Group at least 5 and a maximum of 10 of these questions in each 1 hour evaluation test without reference.

Choose different type questions in the preparation of each test, changing the parameter and data . base to avoid collages

Each question is worth between 10% and 20% depending on the number of questions selected

**Issue:**

1 - in the body R to ∀A, b, c∈ℝ prove that a + c = b + c ⇒a = b (cut law sum)

response:
the reciprocal of the apparent “good definition” of the sum: for any a, b, cR, it follows that if a = b then a + c = b + c .

So if a + c = b + c, then

a + c + (- b) = b + c + (- b)

a + (c + c) = b + (c + c)

a = b

2 - Problem: In R body to ∀A, b, c∈R prove that a = b⇒a⋅c = b⋅c, any c

if= 0 then c = 0 a⋅c = b⋅c

If c ≠ 0 then c⁻¹∈ℝ ec⋅c⁻¹=1

a=b
\[ \Rightarrow a \cdot (c \cdot c^{-1}) = b \cdot 1 \]
\[ \Rightarrow (a \cdot c) \cdot c^{-1} = b \]
\[ \Rightarrow (a \cdot c) \cdot c^{-1} \cdot c = b \cdot c \]
\[ \Rightarrow (a \cdot c) \cdot (c^{-1} \cdot c) = b \cdot c \]
\[ \Rightarrow a \cdot c = b \cdot c \]

3 - Problem: In the body $\mathbb{R}$ to $\forall A, b, c \in \mathbb{R}$ prove that: $a \cdot c = b \cdot c \Rightarrow a = b$ only if $c \neq 0$

If $c \neq 0$ then $c \in \mathbb{R}$, $e \cdot c^{-1} = 1$

\[ a \cdot c = b \cdot c \]
\[ \Rightarrow a \cdot c \cdot c^{-1} = b \cdot c \cdot c^{-1} \]
\[ \Rightarrow a \cdot (c \cdot c^{-1}) = b \cdot (c \cdot c^{-1}) \]
\[ \Rightarrow a \cdot b = \]

4 - Problem: In the body $\mathbb{R}$ to $\forall A, b, c \in \mathbb{R}$ prove that $x = ba$ is the only solution of $a + x = b$

If $a \neq 0$ then $a^{-1} \in \mathbb{R}$, $eb \cdot a^{-1} = (b / a)$

\[ x = ba \]

5 - Problem: In the body $\mathbb{R}$ to $A, b, c \in \mathbb{R}$ prove: If $a \neq 0$ then $y = (b / a)$ is the only solution $a \cdot y = b$

If $a \neq 0$ then $a^{-1} \in \mathbb{R}$, $eb \cdot a^{-1} = (b / a)$

\[ x = ba \]

6 - Problem: In the body $\mathbb{R}$ to $\forall A, b, c \in \mathbb{R}$ prove that: If $A \cdot B = 0$ then $a = 0$ or $b = 0$

Let $A \cdot B = 0 \Rightarrow a \neq 0$ then $a^{-1} \in \mathbb{R}$ and

\[ a^{-1} \cdot (A \cdot B) = a^{-1} \cdot 0 \]
\[ \Rightarrow (a^{-1} \cdot a) \cdot b = 0 \]
\[ = 0 \Rightarrow b \]
tob ≠ 0 the statement is identical

7 - Problem: In the body R to ∀A, b, c∈ R proving that: - (- a) = a

- (- a) + (- a) = 0 = a + (- a) logo - (- a) =

8 - Problem: In the body R to ∀A, b, c∈ R prove that: -(a + b) = (- a) + (- b) = - ab

- (a + b) + (a + b) = 0 = (- a) + a + (- b ) = b = (- a) + (- b) + (a + b) shortly -(a + b) = (- a) + (- b)

- ab = (- a) + (- b)

9 - problem: In the body R to ∀A, b, c∈ℝ prove : -(A⋅B) = (- a) = ⋅b a⋅ (b)

- (A⋅B) + (A⋅B) = 0 = 0⋅b = ((- a) + a) ⋅b = (- a) ⋅b + (A⋅B) logo -(A⋅B) = (- a) ⋅b

10 - Problem: In the body ℝ to ∀A, b, c∈ℝ prove that: (- a) ⋅ (b) = A⋅B

(-a) ⋅ (b) - (- a) ⋅ (b) + (a) ⋅b = (- a) ⋅ (- b) + a ⋅b = 0 = (A⋅B) - (A⋅B) logo (-a) ⋅ (b) = A⋅B

11 - Problem: In the body ℝ to ∀A, b, c∈ℝ prove that: If b ≠ 0 then - (b⁻¹) = (- b) ⁻¹

If b ≠ 0 then b⁻¹∈ℝ and bb ⁻¹ = 1

Induction:

b⁻¹ = (b⁻¹) ¹ = b⁻¹

Assumindo b⁻ⁿ=(b⁻¹)ⁿ

b⁻⁽ⁿ⁺¹⁾=b⁻ⁿ⁻¹=b⁻ⁿb⁻¹=(b⁻¹)ⁿb⁻¹=(b⁻¹)ⁿ⁺¹

12 - Problem: In the body ℝ to ∀A, b, c∈ℝ prove that: (a / b) ≠ 0 if and only if a ≠ 0

If (a / b) ≠ 0 then ((a / b))⁻¹ = ((b / a)) ∈ R immediately ≠ 0

Suppose ≠ 0∧ (a / b) = 0 then a⁻¹ ((a / b)) 0⇔ = (1 / b) = 0 contradicting the fact that 0 has no inverse

Therefore ¬ (a ≠ 0∧ (a / b) = 0) ≠ 0∧ ⇒a (a / b) ≠ 0

13 - Problem: In the body ℝ to ∀A, b, c∈ℝ prove that: (a / b) ≠ 0 if and only if a ≠ 0

If (a / b) ≠ 0 then ((a / b))⁻¹ = ((b / a)) ∈ R immediately ≠ 0

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Therefore ¬ (a ≠ 0∧ (a / b) = 0) ≠ 0∧ ⇒a (a / b) ≠ 0

14 - Problem: In the body ℝ to ∀A, b, c∈ R prove that: (b / b) = 1 for any b ≠ 0

If b ≠ 0 then b⁻¹∈ℝ eb⋅b⁻¹

15 - Problem: In the body ℝ to ∀A, b, c∈ℝ prove that: (a / b) ± (c / d) = (((± adbc)) / (bd)) to b, d ≠ 0

if b, d ≠ 0 then b⁻¹∈ℝ eb⋅b⁻¹ d⋅d⁻¹ = = 1

(((ad ± bc)) / (bd)) = (((ad ± bc)) / (bd)) (bd) (bd)⁻¹
\[= (ad \pm bc) (d^{-1}b^{-1})\]
\[= ad (d^{-1}b^{-1}) \pm bc (d^{-1}b^{-1})\]
\[= \pm ab^{-1} cd^{-1}\]
\[= (a / b) \pm (c / d)\]

16 - Problem: In the body \(R\) to \(\forall A, b, c \in \mathbb{R}\) prove that: \((a / b) \cdot (c / d) = (((c)) / (bd))\), to \(b, d \neq 0\)

If \(b, d \neq 0\) then \(b^{-1}, d^{-1} \in \mathbb{R}\) 
\[= eb \cdot b^{-1} d \cdot d^{-1} = 1\]
\[(c) / (bd) = (c) / (bd) (bd) (bd)^{-1}\]
\[= ac (bd)^{-1}\]
\[= c (d^{-1}b^{-1})\]
\[= (ab^{-1}) (cd^{-1})\]
\[= (a / b) \cdot (c / d)\]

17 - Problem: In the body \(R\) to \(\forall A, b, c \in \mathbb{R}\) prove that: \(If (c / d) \neq 0 then ((a / b) / (c / d)) = ((a \cdot d) / (b \cdot c))\)

If \(b\) and \(d \neq 0\) then \(b^{-1}, d^{-1} \in \mathbb{R}\) 
\[= eb \cdot b^{-1} d \cdot d^{-1} = 1 =\]
\[=((a / b) / (c / d)) = (a / b) (c / d)^{-1}\]
\[= (ab^{-1}) (cd^{-1})^{-1}\]
\[= (ab^{-1}) (c^{-1}d)\]
\[= (d) (b^{-1}c^{-1})\]
\[= (d) (bc)^{-1}\]
\[= ((ad) / (bc))\]

18 - Problem: Find the equivalent measured in radians

(a) \(60 = (\pi / 3)\)
(b) \(135 = (3\pi / 4)\)
(c) \(210^\circ = ((7\pi)/6)\)
(d) \(-150^\circ = -((5\pi)/6)\)
(e) \(20^\circ = (\pi / 9)\)
(f) \(450^\circ = ((5\pi)/2)\)
(g) \(-75^\circ = -((5\pi)/(12))\)
(h) \(100^\circ = ((5\pi)/9)\)
19 - Problem: Find the equivalent measure in degrees to the values given in radians

(a) \( \frac{1}{6} \pi = 30 \)
(b) \( \frac{3}{4} \pi = 135 \)
(c) \( \frac{4}{3}\pi = 240 \)
(d) \(-5\pi = -900 \)
(e) \( \frac{1}{3} = \frac{180}{3\pi} \)
(f) \(-5 = -\frac{5 \times 180}{\pi} \)
(g) \( \frac{11}{12}\pi = \frac{165}{2} \)

20 - Problem: Determine the exact value of the function

(a) \( \sin \left( \frac{4}{3} \pi \right) = -\frac{1}{2} \sqrt{3} \)
(b) \( \cos \left( -\frac{1}{6} \pi \right) = \frac{1}{2} \sqrt{3} \)
(c) \( \sin 7\pi = 0 \)
(d) \( \cos \left( -\frac{5}{2} \pi \right) = 0 \)

21 - Problem: Use periodicidade and values in the range \( 0 \leq t \leq 2\pi \) to determine the exact value of the function:

(a) \( \sin \left( -\frac{5}{4} \pi \right) = \frac{1}{2} \sqrt{2} \)
(b) \( \cos \left( -\frac{5}{4} \pi \right) = -\frac{1}{2} \sqrt{2} \)
(c) \( \sec \left( -\frac{5}{4} \pi \right) = -\sqrt{2} \)
(d) \( \csc \left( -\frac{5}{4} \pi \right) = \sqrt{2} \)

22 - Problem: Use periodicity and values in the range \( 0 \leq t \leq 2\pi \) to determine the exact value of the function:

(a) \( \sin \left( \frac{7}{2} \pi \right) = -1 \)
(b) \( \cos \left( \frac{5}{2} \pi \right) = 0 \)
(c) \( \sec \left( \left( \frac{11}{2} \right) \pi \right) \) not determined
(d) \( \csc \left( \frac{9}{2} \pi \right) = 1 \)

23 - Problem: Use periodicity and values in the range \( 0 \leq t \leq 2\pi \) to determine the exact value of the function:

(a) \( \tan \left( \frac{4}{3} \pi \right) = \sqrt{3} \)
(b) \( \cot \left( \frac{4}{32} \pi \right) = 1 \sqrt{2} + \)
(c) \( \tan \left( -\frac{1}{6} \pi \right) = -\frac{1}{3} \sqrt{3} \)
(d) \( \cot \left( -\frac{1}{6} \pi \right) = -\sqrt{3} \)
24 - Problem: Find all values of t in the interval [0, 2π]

(a) \( \sin t = -1, t = \left( \frac{3\pi}{2} \right) \)
(b) \( \cos t = 1, t = 0 \)
(c) \( \tan t = -1, t \in \left( \frac{3\pi}{4}, \frac{7\pi}{4} \right) \)
(d) \( \csc t = 1, t = \frac{\pi}{2} \)

25 - Problem: Find all values of t in the interval [0, 2π]

(a) \( \sin t - \frac{1}{2} \), \( t \in \left( \frac{7\pi}{6}, \frac{11\pi}{6} \right) \)
(b) \( t = \cos \left( \frac{1}{2} \right) \), \( t \in \left( \frac{\pi}{3}, \frac{5\pi}{3} \right) \)
(c) \( \cot t = 1, t = \frac{\pi}{4} \)
(d) \( t = 2 \sec, t \in \left( \frac{\pi}{3}, \frac{5\pi}{3} \right) \)

26 - Problem: Find all values of t in the interval [0, 2π]

(a) \( \sin t = \frac{1}{2}, t \in \left( \frac{\pi}{4}, \frac{3\pi}{4} \right) \)
(b) \( \cos t = -\frac{1}{2}, t \in \left( \frac{3\pi}{4}, \frac{5\pi}{4} \right) \)
(c) \( \tan t = -\frac{1}{3}, t = \frac{11\pi}{6} \)
(d) \( \cot t = \frac{1}{3}, t \in \left( \frac{\pi}{3}, \frac{4\pi}{3} \right) \)

27 - Problem: Find \( \tan \theta \) is the angle between the straight lines with inclinations given

(a) \( \frac{1}{2} \) and \( -\frac{3}{4} \)
\[ \arctan \left( \frac{1}{2} \right) = 0.46365 \] \( \approx 26.565 \)
\[ \arctan \left( -\frac{3}{4} \right) = -0.6435 \] \( \approx -36.870 \)
\[ 26.565 - (-36.870) = 63.435 \]
\[ \tan(63.435) \approx 2.0 \]

(b) \( \frac{2}{7} \) and \( \frac{7}{2} \)
\[ \arctan \left( \frac{2}{7} \right) = 0.27830 \] \( \approx 15.945 \)
\[ \arctan \left( \frac{7}{2} \right) = 1.2925 \] \( \approx 74.055 \)
\[ 74.055 - 15.945 = 58.110 \]
\[ \tan(58.110) \approx 1.6072 \] \( \approx (8/5) \)

28 - Problem: Find \( \tan \theta \) is the angle between the straight lines with inclinations given

(a) \( -\frac{3}{5} \) and 2
\[ \arctan \left( -\frac{3}{5} \right) = -0.54042 \] \( \approx -30.964 \)
\[ \arctan(2) = 1.071 \] \( \approx 63.432 \)

26
63. \( \tan(94.396 \, ((\pi / (180)).)) = -13.008 \approx -((13) \, / \, (100)) \)

(b) \(-1/3\) and \(-1/10\)

\(\arctan(-1/3) = -0.32175 \approx -18.435\)

\(\arctan(-1/10) = -9.9669 \times 10^{-2}\)

\(\theta = (-9.9669 \times 10^{-2}) - (-18.435) = 18.335\)

\(\tan(18.335 \, ((\pi / (180))).) = 0.33140 \approx (1/3)\)

29 - Problem: Find to the nearest 1° to measure the angle between the straight lines with the given inclinations

(a) -3 and 2

\(\arctan(-3) = -1.249 \approx -71.562\)

\(\arctan(2) = 1.1071 \approx 63.432\)

\(\theta = 63.432 - (-71.562) = 134.99\approx 135° \) (complementary 180-135 = 45)

(b) \((3/2)\) and \((1/4)\)

\(\arctan((3/2)) = 0.98279 \approx 56.310\)

\(\arctan((1/4)) = 0.24498 \approx 14.036\)

\(\theta = 56.310 \approx 42.274\approx 42°\)
Summary

In this unit has carried out a review and an assessment of the knowledge of the basic prerequisites of the calculation, including algebra (operations on numerical sets N, Z, Q and R) and trigonometry.

In the first part of this activity the numbers were treated, operations on numerical sets, inverse operations, operations not defined in R.

In the second part treated the unit circle and the definition of radian as a measure of an angle, trigonometry functions and the trigonometric identities.

Unit Evaluation

Check your understanding

Unit summative evaluation

Instructions

Group a minimum of 5 and a maximum of 10 of the following questions in each evaluation test
1 hour, without consultation

Choose different type questions in the preparation of each test, changing. the parameters and database to avoid collages

Evaluation Criteria/Grading scheme

Each question is worth between 10% and 20% depending on the number of selected questions 100;

Total points awarded is

Results between 90-100 is excellent,
result from 70-89 is good;
Result between 50-69 is enough,
less than 50 results is inadequate;

Rating
Solve for the indicated variable, if you can not tell why:

\[ 12 - [3 + 4m - 6(3m - 2)] = -7(2n - 8) - 3[(m - 2) + 3m - 5] \]

\[ 12 - [3 + 4m - 6(3m - 2)] = -7(2n - 8) - 3[(m - 2) + 3m - 5] \]

\[ 2m - \frac{5m}{6} = \frac{3m}{72} + \frac{4}{5}m - \frac{5m}{3} = \frac{3m}{72} + \frac{4}{5} \]

\[ \sqrt{m + 1} - 4 = \sqrt{m + 1} - 4 \]

\[ \frac{(m + 1)^{1/2}}{4} = (m + 1)^{1/2} = -4 \]

Applies logarithm property to simplify expressions

\[ a) \log_3 3^7 \log_3 3^7 \]

\[ b) \log_3 3^{2n} \log_3 3^{2n} \]

\[ c) \log_3 (3^{2n}) \cdot 3^{2n} \]

Write each expression as a single logarithm:

\[ a) 2\log_b u + 3\log_b v \]

\[ b) \ln \sqrt{x - 1} + \ln (x + 1 - 2ln(x^2 - 1)) \]

Use the unit circle to find all exact values that make true the given equation, the indicated ranges:

\[ a) \cos \theta = \frac{\sqrt{3}}{2} 0 \leq \theta \leq 2\pi \cos \theta = \frac{\sqrt{3}}{2} 0 \leq \theta \leq 2\pi \]

\[ b) \sin \theta = -1.0 \leq \theta \leq 4\pi \sin \theta = -1.0 \leq \theta \leq 4\pi \]

\[ c) \tan \theta = -1.0 \leq \theta \leq 2\pi \tan \theta = -1.0 \leq \theta \leq 2\pi \]

Reduce the trigonometric function given to an equivalent function with argument in the first quadrant,

\[ \leq \theta \leq \pi/2 \]

\[ a) \cos(3\pi - \theta) \cos(3\pi - \theta) \]

\[ b) \sin(6\pi + \theta) \sin(6\pi + \theta) \]

\[ c) \tan \left(\frac{\pi}{2} - \theta\right) \tan \left(\frac{\pi}{2} - \theta\right) \]

Say it is true or false the following statement: “the solution set of the equation

\[ x = \frac{1}{1/x} \] is the set of all real numbers.”

The doctor prescribed a dose of 600 milligrams of amoxicillin. The pharmacy has only one amoxicillin suspension with a concentration of 125 milligrams per 5 milliliters. How much liquid suspension should be given to the patient?
Answers

m = 2

m is not full

is not possible (square root is positive)

m = -65

m = -65

7

1/9

-19 / x^4

a. \( \log_{b} uv^3 \)

b. \(-\frac{3}{2} \ln(x^2 - 1) - \frac{3}{2} \ln(x^2 - 1) \)

a. \( \frac{\pi}{6} \) and \( \frac{\pi}{2} \)

b. \( \frac{\pi}{2} \)

c. \( \frac{\pi}{11} / 6 \) 3/2 3/4 and \( \frac{\pi}{2} / 4 \)

a. \( \cos(\theta) \cos(\theta) \)

b. \( \sin(\theta) \sin(\theta) \)

c. \( \tan\left(\frac{\pi}{2} - \theta\right) \tan\left(\frac{\pi}{2} - \theta\right) \)

False (except 0)

24 ml

Readings and Other Resources

Readings and other features of this unit are in the list of “Readingsand other course resources”. 
Unit 1. Functions and limits functions

Introduction to the Unit

This unit introduces the basic elements of $\mathbb{R}$ topology, presents the elementary real functions and develops methods for solving equations and inequalities in $\mathbb{R}$, $\mathbb{N}$. Also introduces functions for $\mathbb{R}$ designated sequences, studies the convergence of sequences in $\mathbb{R}$, and finally studies limits of real functions either in real points or at infinity.

Unit Objectives

By the end of this unit, you should be able to:

- determine domain and elementary real functions;
- solve equations and inequalities involving elementary real functions, in particular the absolute value and determine the solution ranges;
- draw the graph of elementary real functions;
- operate limits of sequences;
- operate limits of real functions.

Key Terms

**Less than**: order relation in $\mathbb{R}$

**Plus**: conjunction with upper

**Supreme bounds**: lower upper bounds of the

**Range**: subconjunction of $\mathbb{R}$ that has any point between its points.

**Straight finished**: $\mathbb{R}$ together with the symbols

**Variable (var.)**: Indefinite appointment of elements in a set;

**Var. independent**: free to take arbitrary values in a set;

**Var. dependent**: value resulting from the calculation of an analytical expression;

**Point function**: ordered pair $(x, f(x)) \in A \times B$
**Function value:** appointment of an element \( f(x) \in B \)

**Codomain:** set of arrival of the functional relationship,

**Image:** \( \text{cj. all function values} \);

**Graph:** \( \text{cj. all function points} \);

**Fun. real:** domain and finish set are subconjuntos of \( R \);

**Abscissa:** horizontal axis in the plane \( R^2 \);

**Ordinate:** vertical axis in the plane \( R^2 \);

**Translation:** chart displacement change of variable;

**Fun. pair:** property \( f(x) = f(-x) \)

**Fun. odd:** property \( f(x) = -f(-x) \)

**Fun. periodic:** Property \( f(x) = f(x + T) \)

**Period:** low \( T \) a function. periodic,

**Fun. constant:** when \( y = c \ \forall x \in \text{dom} f \), where \( c \) is a constant;

**Fun. zero:** when \( y = 0, \ \forall x \in \text{dom} f \)

**Fun. Almost zero:** zero except in \( \text{cj. countable domain points} \),

**Zerof** solution \( (x) = 0 \)

**Cj. zeros:** \( \text{cj. the domain points whose image is zero} \);

**Cjzero:** \( \text{f intersection with the zero function 0} \)

**Fun. identity:** the image is the very independent variable \( f(x) = x \)

**Fun. linear:** fun. the form \( f(x) = mx \)

**Fun. order:** fun. the form \( f(x) = mx + b \)

Fun. quadratic: fun. the form \( f(x) = bx + c + ax^2 \)

**Grade:** greatest exponent of \( x \) in \( f \) expression \( (x) \)

**Coefficients:** constant multiplicative terms of the polynomial;
**Straight**: one chart function order:

**Slope**: slope of the line to the horizontal,

**Intersection-y**: ordinate of the straight line intersects the vertical axis;

**Discriminant**: expression involving the coefficients of function quadratic

**Fun. rational**: $f(x)$ a ratio of polynomials

**Vertical asymptotes**: vertical line $x = a$ in points for annulment of the denominator;

**Fun. With radicals**: $f(x)$ with radicals;

forms to $+\sqrt{2}$ and $-\sqrt{b}$

**Vizinhança**: points at a distance less than

**Adhering conjugates**: any neighborhood intercepts conj

build any neighborhood intercepts conj. except himself point: open equal to its inside: locked equal to its closing;

**Limit at infinity**: the function will possibly any L neighborhood if $x \to +\infty$

**Limit in point**: the function will possibly any L neighborhood if $x \to$ the

**Infinite limit**: there is no real number to limit property;

**Horizontal asymptote**: the line $y = L$, where $L$ is a limit at infinity,

**Asymptotic functions**: the difference tends to zero

**Oblique asymptote**: a straight line $y = mx + b$ asymptotic $f(x)$

**Lateral limits**: the limit as $x$ tends to a point laterally;

**Equivalencia asymptotic**: functions in the same equivalence class generated by the asymptotic order 1.1.
Learning activities

**Activity 1 - Order Properties of \( \mathbb{R} \)**

**Introduction**

To be introduced to the \( \mathbb{R} \) order property, the property of the supreme and deduct are some consequences introduction.

the finished line with the symbols +, and operations and exceptions to these symbols are given some topological concepts \( \mathbb{R} \) for points and notable subsets of \( \mathbb{R} \).

**Activity Details**

\( \mathbb{R} \) can establish an order \((<)\) called “less than” and the reverse order \( (>)\) called “greater than” as follows: given two real numbers \( a, b \) only one of the conditions may be true:

\[
\begin{align*}
    a < b & \quad \forall a, b \\
    a < b & \quad \forall a, b \\
    a \leq b & \quad \forall a, b \\
    a \leq b & \quad \forall a, b \\
\end{align*}
\]

0. define \( \leq \) a tin order “less or equal” to be equivalent to smaller \(<\) or equal \(=\).

the geometric model \( \mathbb{R} \) is a set of points on a continuous and graded line called line real:

\[
\begin{array}{c}
\mathbb{R} \\
\hline
-3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\end{array}
\]

a subset of \( \mathbb{R} \) a is increased if \( \exists b \in \mathbb{R}, \forall x \in A \exists b \in \mathbb{R}, \forall x \in A, x \leq b \); the b element is called upper bound of A. similar definition is given for lower bound of A.

A subset of \( \mathbb{R} \) A is bounded has upper bound and lower bound (using the term “bounded” instead of “limited” to not overload this last)

Ex: \([2, + \infty \] has lower bounds but is not limited; 4 is an upper bound of] -3.3 \]; -2 is a lower bound of \([1, + \infty \]

[The Smallest upper bound is called supreme \( A \in \mathbb{R} \) subset and the greatest lower bound is called infinitesimal subset \( A \in \mathbb{R} \)

Ex: is supreme \( A \) \( x \leq \beta \) iff, \( \forall x \in A \) and if \( y < \beta \) then \( y \) is no upper bound of \( A \).

the set has the \( \mathbb{R} \) property supreme, or any subset \( A \subset \mathbb{R} \) defined above has upper bound (similarly for the other cases delimiting)

a interval \( \mathbb{R} \) is a subset \( I \subset \mathbb{R} \subset \mathbb{R} \) with the following property (convexity):

\[
x, y, \in I \quad x \leq y \Rightarrow \exists I, x, y, \in I \quad x \leq y \Rightarrow \Rightarrow I
\]

Note that \( x \leq \to y \leq \to \leq y \) means \( x \leq \to y \leq \to \leq y \)
A range can be

- **open**: \((a, b) = \{x \in \mathbb{R}: a < x < b\}\) (or \(a, ba, b\))
- **closed**: \([a, b] = \{x \in \mathbb{R}: a \leq x \leq b\}\) (or \([a, b]\))
- **semi-open**: \(\{a, b\} = \{x \in \mathbb{R}: a < x \leq b\}\) and \([a, b) = \{x \in \mathbb{R}: a \leq x < b\}\)

points \(a, ba, b\) are called edges or ends of the range

the \(\mathbb{R}\) together as subset of itself, is not bounded, ie, it has no upper bound or lower bound properties:

So it is convenient to introduce two symbols \(-\infty, +\infty\) with the following properties:

- any real number is higher than \(-\infty\);
- any real number is less than \(+\infty\),

and define the finished line as the "closing" of the real line:

\[
\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\} = \mathbb{R} \cup \{-\infty, +\infty\}
\]

the \(\mathbb{R}\) and set \(\mathbb{R} \mathbb{R}\) can be written as range: \(\mathbb{R} = \infty (-, +\infty) \mathbb{R} = \infty (-, +\infty)\)

and \(\mathbb{R} = \infty [-, +\infty\] \mathbb{R} = \infty [-, +\infty\]

Note that the symbols \(\infty + \infty\infty + \infty\) are not real numbers, it does not meet all the real numbers properties, \(\mathbb{R} \mathbb{R}\) ..., is not an algebraic body)

Some operations left undefined in \(\mathbb{R}\) can now be defined in \(\mathbb{R}\):

\[
\frac{1}{0} = +\infty, \frac{-1}{0} = -\infty, \frac{1}{0} = +\infty, \frac{-1}{0} = -\infty
\]

and other operations can be defined in \(\mathbb{R}\):

\[
\begin{align*}
\frac{1}{+\infty} &= \frac{1}{0} = 0 \\
\frac{1}{-\infty} &= \frac{1}{0} = 0 \\
+\infty + \infty &= +\infty + \infty = +\infty \\
-\infty + \infty &= -\infty + \infty = -\infty \\
+\infty \times (+) &= +\infty \times (+) = +\infty x \in R \\
-\infty \times (+) &= -\infty \times (+) = -\infty \\
x + \infty &= +\infty x \in R \\
x \times (+) &= +\infty x \times (+) = +\infty \text{ if } x > 0 \\
x \times (+) &= -\infty x \times (+) = -\infty \text{ if } x < 0 \\
\end{align*}
\]
However, some operations remain undefined in $\mathbb{R}$:

$$\infty \times 0 \equiv \infty, 0 \times \infty = \frac{0}{\infty} \equiv 0 \mathbb{R}^\infty \equiv 0, 0 \mathbb{R}^\infty \equiv 0, 0 \equiv 0.$$

These operations are undefined in $\mathbb{R}$ because they have a single result, however, may have any results, depending on each case. Later we will know how to determine these values for each case (lift indeterminacies).

The distance between two points on the straight line, $b \mathbb{R}$ is given by the absolute value of the difference between the two numbers, $|ab|$.

The metric properties of the distance between two points $a, b \in \mathbb{R}, b \in \mathbb{R}$ points is:

- $|ab| \geq 0, \forall a, b \in \mathbb{R}, a, b \in \mathbb{R}$
- $|ab| = 0 \iff a = b$
- $|AB| = |bA|
- $|AB| \leq |aC| + |cb| \in \mathbb{R} \in \mathbb{R}$

The distance $x \mathbb{R}$ any point zero is given by the absolute value $|0-x| = |x|$

The set of points that are distant for the distance less than is denoted by vizinhança-ε:

$$V_\{\varepsilon\} (a) = \{x \in \mathbb{R} : |x| < \varepsilon\}$$

and corresponds to an open interval center $a$ and radius $\varepsilon$:

$$V_\{\varepsilon\} (a) = \{a-\varepsilon, a + \varepsilon\}$$

And is a position outside and if there a neighborhood of the included the complement of $E$: If there $\varepsilon > 0$ such that $V_\{\varepsilon\} (a) \subset E$, then $E$ is a point of the edge $E$ is neither inside nor outwardly.

Is an adherent point $E$ any neighborhood to have an element of $E$: any $\varepsilon > 0$ $\varepsilon \notin V_\{} (a) \cap E \neq \emptyset$ is a point of accumulation $E$ in the neighborhood to have any an element different from the $E$: Any $\varepsilon > 0$, $(V_\{\varepsilon\} (a) - \{a\}) \neq \emptyset \cap E$

A is an isolated point of $E$ iff $\{a\}$ is a neighborhood of $a$, for any $\varepsilon > 0$

Ex: the difference between adherent points and accumulation points is isolated points any accumulation point is an adherent point but there are sticky points that are not accumulating points: isolated points $E \subset \mathbb{R}$.

Remarkable Subsets

iNTE: Interior And is the set of all interior points of $E$
eXTE Outdoor E, is the set of all exterior points and
∂E: maple And is the set of all points of the board of E
E: closing And is the set of all adherent points E
E ‘: derived from E, is the set of all Accumulation of points
characterization of some subset
E is open iff all its points are interior to E
E is closed iff your E ^ complement (c) is open.
it is dense in R iff \( E \subset R \) (the clasp is all R)
and is rare in R iff Int =
E is perfect if closed without isolated points.

Conclusion

In this activity, R unit was characterized as a complete ordered field with the property of the
supreme symbols. Any other set with the same properties is isomorphic to R was defined the
finished line with the introduction of \(-, 0, \infty, 0, \infty, +,\) and operations and exceptions with
these symbols (indeterminate). terminology were made on the notable points and subsets of R.

Evaluation

Group at least 5 and a maximum of 10 questions in each of the following evaluation test of 1
hour without consultation.

choose different type in question . preparation of each test by changing the parameters and
database to avoid collages

each question is worth between 10% and 20% depending on the number of selected questions
1.2.

Activity - functions in R

Introduction

This unit introduces basic functions in R and calculates its domains resolving certain inequalities.
Also presents in some detail the inequalities resolution method by between zeros functions
signal study or discontinuity points.

Activity Details

A function is a complete and univocal relation of domain to a codomain B.

Total because all elements of the domain a must be related to some element of the codomain B
and univocal because any domain element a can only be related to a single element of the
codomain B.
a function to indicate objectively how the domain elements \( x \in ax \in a \) relate to the elements of the codomain \( y \in y \in B \). The way “classic” to define a function is the functional formula

\[
y = fy = f(x), \text{where } f(x)f(x)
\]

is an analytical expression in the variable \( x \).

the domain of a function in \( \mathbb{R} \), when not specified, is defined as the largest subset of \( \mathbb{R} \) where the analytical expression \( f(x)f(x) \) has significance to the operations defined in \( \mathbb{R} \).

it is recalled exceptions already mentioned in previous units for operations of \( \mathbb{R} \) are:

- division by zero,
- zero power base of zero;
- root extraction couple of negative numbers,
- logarithms of any basis for not positive real numbers,
- negative basis of logarithms or base equal to 1.

in determining the area, the exceptions give rise to equations or inequations:

Ex: If a function is given by \( y = \sqrt{g}y = \sqrt{g}(x) \), then the domain is the set of all values where

\[
g(x) \geq 0 \quad g(x) \geq 0
\]

a inequations resolution method in \( \mathbb{R} \), ie \( f(x) \geq 0, f(x) \geq 0 \), is:

- find the zeros and points of discontinuity of the analytical expression \( f(x)f(x) \) sign;
- study the sign \( f(x)f(x) \) between the zeros and / or discontinuities, because these are the only places where a real function can change
- determining the intervals at which the function signal satisfies the inequality
- recall up some properties of elementary functions:

**polynomial** \( f(x)f(x) \)

\[
f(x) = ax + b, \neq 0:f(x) = ax + b, \neq 0 \text{ root (or zero) is } -\frac{b}{a} \quad \text{ and signal to: (}+,-\text{) to: (}+,-\text{)} \text{ informs it about the function of the signal: before the root of }
\]

\[
f(x) = ax^2 + bx + c, \neq 0:f(x) = ax^2 + bx + c, \neq 0 \text{ signal discriminating } \Delta = b^2 - 4ac \Delta = b^2 - 4ac \text{ tells you if there is root in } \mathbb{R} \text{ and solving formula (or notable cases) provide the value of the roots. The signal (}+,-\text{) to: (}+,-\text{) informs the function signal: (}+,+\text{)(}+,-\text{) between before and roots. When there is only one zero signal before and after aresame. For cubic and higher orders, estimate a zero and factoring to FIND others. The number of zeros in } \mathbb{R} \text{ is always equal or less than the degree of the polynomial. Anyway the fundamental theorem of algebra guarantees at least } n \text{ raisex a polinokial equation of degree } n
\]

**rational**: \( \frac{f(x)f(x)}{g(x)g(x)} \) where \( f(x)g(x)f(x)g(x) \) are polynomials
there are zeros when the dividend is null (applies if the previous case)

there is discontinuity when the divisor is zero (applies the previous case)

radical: \( \sqrt[n]{g(x)} \)\( \sqrt{g(x)} \)

root extraction is poteciação by factions,

calculate the domain \( x \) as the largest subset of \( \mathbb{R} \) for which the expression is defined, restricted \( g(x) \geq 0 \) \( g(x) \geq 0 \) when is even (when \( n \) is odd, there is no such restriction).

in seeking of zeros is tried to isolate the end with the moiety on one side of the equation, to raise the two sides of power. \( \text{NoNc} \) (the result must be checked for values may arise which are not part of the solution)

in case of fraction with denominator of the type \( \sqrt{b} \), is multiplied by \( (a - \sqrt{b})(a + \sqrt{b}) / (a - \sqrt{b})(a + \sqrt{b}) \) provides that \( b \neq a^2 \) \( b \neq a^2 \) (note that this multiplication is always possible because we are in fact multiply by +1)

absolute value: \( | g(x) | \)

by definition, the absolute value is equivalent to the following disjunction of conjunctions:

\[
\begin{align*}
| g(x) | &= g(x) \land g(x) \geq 0 \lor (-g(x) \land g(x) < 0) \\
| g(x) | &= g(x) \land g(x) \geq 0 \lor (-g(x) \land g(x) < 0)
\end{align*}
\]

Note: do not confuse braces of linear algebra (and) with module definition of chavetas (or),

replace the absolute value by simple expressions and apply the previous case

if the function has multiple expressions with absolute value, the method is applied to each one of the expressions and the result is the combination in the overlapping regions

systematic work in an appropriate signal frame.

exponential:

the domain only depends on \( g(x) \)

has no roots and is always positive

when the \( base > 1; base > 1 \) function is monotone increasing and preserves the order:

\[
1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2), \quad 1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)
\]

with the base between zero and one, \( 0 < a < 1,0 < a < 1 \) the function reverses the order has.

When \( a = e \), the special case of natural exponent

logarithm (natural): \( \log_b g(x) \)

the base \( b \) must be positive and other than 1

\( \log_b g(x) \log_b g(x) \) is defined only in the domain

\( 0 < g(x) < +\infty, 0 < g(x) < +\infty \)
has zero \( g(x) = 1g(x) = 1 \)

is negative (-) where \( 0 < g(x) < 10 < g(x) < 1 \)

is positive (+) to \( g(x) > 1g(x) > 1 \) (incremental):

When \( b = \text{base} \) and has the natural logarithm

trigonometric

\[ g \cos(x)g \cos(x) \] has roots \( g(x) = n\pi, g(x) = n\pi, n \in \mathbb{Z} \), and function value ranges from -1 to +1;

\[ g \sin(x)g \sin(x) \] has roots

\[ g(x) = (n - (1/2))\pi, n \in \mathbb{Z} \]

and the function value varies between -1 and +1;

\[ \tan g(x) = \frac{(\sin g(x))}{(\cos g(x))} \tan g(x) = \frac{(\sin g(x))}{(\cos g(x))} \] has zeros in

\[ g(x) = (n - (1/2))\pi, g(x) = (n - (1/2))\pi, n \in \mathbb{Z} \], and has discontinuities in

\[ g(x) = n\pi, n \in \mathbb{Z} \]

is always growing between “lips”

Conclusion

A function is a univocal relation of a domain to a codomain and functional formula \( y = f(x)\) relates elements \( x \) domain elements \( y \) codomain.

the determination domain of a function \( \mathbb{R} \) may give rise to inequalities or equations that are restrictive conditions on the analytical expression \( f(x)f(x) \) of a function; .. a of the resolution of a first inequality methods is zeros or discontinuities analytical expression and then studies the sign of this expression between zeros or discontinuities.

the basic analytic expressions are polynomials, rational, with radical, exponential and logarithmic.

Evaluation

Group a minimum of 5 and a maximum of 10 of the following questions in each evaluation test . 1 hour without consultation. Choose different type questions in the preparation of each test by changing the parameters and database to avoid collages each question is worth between 10% and 20% depending on the number of selected question

Problem

Determine domain, image and function of the graphic outline \( G(x) = x^2 \)

5-A:

\[ \text{Dom}(G) = \mathbb{R} \]

\[ \text{Im}(G) = [-\infty, 5] \]

Problem: Determine domain, image and function \( g \) graphic outline \( (x) = \sqrt{9-x^2} \)
Answer:

Dom (g) = [ -3, 3]
Im (g) = [0, 3]

Problem: Determine domain, image and function of the graphic outline H (x) = | x-1 |

Reply:

Dom (H) = R
Im (H) = [0, + ∞

Problem: Determine domain, image and function h graphic outline (x) = | x | -1

Answer:

Dom (h) = R
Im (H) = [0, + ∞

Problem: Determine domain, image and function of the graphic outline F (x) = ((4x²-1) / (2x + 1)) = 2x-1

response:

Dom (F) = R - {- (1/2)}
Im (F) = R - {-2}

Problem: Determine domain, image and function g graphic outline (x) = ((x³-3x²-4x + 12) / (x²-x-6)) = x-2

response:

Dom (g) = ] - ∞, -2 [ ∪ -2, 3 [ ∪ ] 3, + ∞
Im(g) = ] - ∞, -4 [ ∪ -4,1 [ ∪ ] 1, + ∞

Problem: Determine domain, image and function graph sketch f (x) = {x²-4 if x ≠ -3, -2 and if x = -3

answer:

Dom (f) = R
Im (f) = ] - 4, + ∞

Problem: Determine domain, image and function graphic outline (x) = {x + 5 if x < -5, √(25-x²) is -5≤x≤5, and x-5 if x> 5

A.

Dom (φ) = R
Im (φ) = R
Problem: Determine domain, image and function of the graphic outline \( g(x) = \begin{cases} 6x + 7 & \text{if } x < -2, \\ 3 & \text{if } x = -2, \\ 4 - x & \text{if } x > 5 \end{cases} \)

A.

\( \text{Dom} (\varphi) = \mathbb{R} \)

\( \text{Im} (\varphi) = [-\infty, 6] \)

Problem: Determine domain, image and function of the graphic outline \( G(x) = \frac{((x^2 + 3x - 4)(x^2 - 5x + 6))}{((x^2 - 3x + 2)(x - 3))} = x + 4 \)

A.

\( \text{Dom} (\varphi) = [-\infty, 1] \cup [1.2] \cup [2.3] \cup [3, +\infty) \)

\( \text{Im}(\varphi) = [-\infty, 5] \cup [5.6] \cup [6.7] \cup [7, +\infty) \)

Problem: Determine domain, image and function \( h \) graphic outline \( (x) = \sqrt{x^2 - 5x + 6} \)

Answer:

\( \text{Dom} (\varphi) = (-\infty, 2] \cup [3, +\infty) \)

\( \text{Im}(\varphi) = [0, +\infty) \)

Problem: Determine domain, image and function graph sketch \( f(x) = \frac{(x^3 + 3x^2 + x + 3)}{(x + 3)} = x^2 + 1 \)

\( \text{Dom} (\varphi) = (-\infty, -3] \cup [-3, +\infty) \)

\( \text{Im}(\varphi) = [1, +\infty) \)

Problem: Determine domain, image and function of the graphic outline \( F(x) = \frac{(x^4 + x^3 - 9x^2 - 3x + 18)}{(x^2 + x - 6)} = x^2 - 3 \)

A.

\( \text{Dom} (\varphi) = (-\infty, -3] \cup [-3, 2] \cup [2, +\infty) \)

\( \text{Im}(\varphi) = [-3, +\infty) \)

Problem: Determine domain, image and function \( g \) graphic outline \( (x) = |x| \cdot |x - 1| \)

response:

Study the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-x</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>x + 1</td>
<td>x + 1</td>
</tr>
<tr>
<td>(</td>
<td>x</td>
<td>\cdot</td>
</tr>
</tbody>
</table>

\( \text{Dom} (\varphi) = \mathbb{R} \)

\( \text{Im} (\varphi) = [0, +\infty) \)
Problem: Determine if the data set is a function \( y = f(x) \). If what your domain?

(a) \( \{(x, y): y = \sqrt{x + 1}\} \)

is a function. \( \text{Dom} f = \{x \in \mathbb{R}: x + 1 \geq 0\} = [-1, +\infty[ \)

(b) \( \{(x, y): y = \sqrt{x^2-1}\} \)

is a function. \( \text{Dom} f = \{x \in \mathbb{R}: x^2-1 \geq 0\} = ]-\infty, -1[ \cup [1, +\infty[ \)

(c) \( \{(x, y): y = \sqrt{1-x^2}\} \)

is a function. \( \text{Dom} f = \{x \in \mathbb{R}: 1-x^2 \geq 0\} = [-1,1] \)

(d) \( \{(x, y): x^2 + y^2 = 1\} \)

is not a function, for endorelação (x, y) does not is univocal, for example ((1/2) ((\sqrt{3}) / 2)) and ((1/2) - ((\sqrt{3}) / 2)) both belong to endorelaçãoProblem.:

Determine whether the data set is a function. if what your domain

(a) \( \{(x, y): y = (x-1)^2 + 2\} \)

is a function. \( \text{Dom} f = \mathbb{R} \)

(b) \( \{(x, y): x = (y-2)^2 + 1\} \)

is not a function, for endorelação (x, y) is not univocal, for example (2,1) and (2.3) both belong to endorelação²-1).

(c) \( \{(x, y): y = (x + 2)\} \)

It is a function. \( \text{Dom} f = \mathbb{R} \)

(d) \( \{(x, y): x = (y + 1)^3 - 2\} \)

is a function. \( \text{Dom} f = \mathbb{R} \)

Problem: Given \( g(x) = 3x^2 - 4 \)-ache

(a) \( g(-4) = 3(-4)^2 = 4 - 44 \)

(b) \( g((1/2)) = 3((1/2))^2 = -4 - ((13) / 4) \)

(c) \( g(x^2) = 3(x^2)^2 = 3x^4 \)

(d) \( g(3x^2-4) = 3(3x^2-4)^2 = 27x^2 - 72x^2 + 44 \)

(e) \( g(xh) = (xh)^2 = 3h^2 - 4hX + 3x^2 - 4 \)

(f) \( g(x) - g(h) = (3(x)^2 - 4) - (3(h)^2 - 4) = 3x^2 - 3h^2 \)

(g) \( ((g(x + h) - g(x)) / h) = (((3(x + h)^2 - 4) - (3(x)^2 - 4)) / h) = 6x + 3h \) for \( h \neq 0 \)
Problem: Being \( f(x) = \sqrt{x} \) and \( g(x) = x^2 + 1 \), determine the functions and the domain of the resulting function:

(a) \( (f + g)(x) = \sqrt{x} + x^2 + 1 \), and Bishop \( f + g = [0, +\infty) \)

(b) \( (gf)(x) = \sqrt{x} \cdot (x^2 + 1) \) and Dom \( f \cdot g = [0, +\infty) \)

(c) \( (fg)(x) = \sqrt{x} \cdot (x^2 + 1) \) and Dom \( f \cdot g = [0, +\infty) \)

(d) \( (f / g)(x) = (\sqrt{x}) / (x^2 + 1) \), Dom \( f / g = [0, +\infty) \)

(e) \( (g / f)(x) = (x^2 + 1) / (\sqrt{x}) \), Dom \( g / f = ]0, +\infty[ \)

Problem: Being \( f(x) = |x| \) and \( g(x) = |x-3| \), determine the functions and the domain of the resulting function:

(a) \( (f + g)(x) = |x| + |x-3| \), and Dom \( f + g = \mathbb{R} \)

(b) \( (fg)(x) = |x| - |x-3| \), and Dom \( f \cdot g = \mathbb{R} \)

(c) \( (fg)(x) = |x| \cdot |x-3| \), and Dom \( f \cdot g = \mathbb{R} \)

(d) \( (f / g)(x) = (|x|) / (|x-3|) \), and Dom \( f / g = \{3\} \mathbb{R}^{-} \)

(e) \( (g / f)(x) = (|x-3|) / (|x|) \), and Dom \( g / f = \{0\} \mathbb{R}^{-} \)

Problem: Being \( f(x) = \sqrt{x + 4} \) and \( g(x) = x^2 - 2 \), determine the functions and the domain of the resulting function:

(a) \( (f + g)(x) = \sqrt{x + 4} + x^2 - 2 \) and Bishop \( f + g = \{x \in \mathbb{R} : x + 4 \geq 0\} = [-4, -\infty) \)

(b) \( (gf)(x) = \sqrt{x + 4} + 2 \cdot x^2 \) and Bishop \( f \cdot g = \{x \in \mathbb{R} : x + 4 \geq 0\} = [-4, -\infty) \)

(c) \( (fg)(x) = \sqrt{x + 4} \cdot (x^2 - 2) \) and Dom \( f \cdot g = \{x \in \mathbb{R} : x + 4 \geq 0\} = [-4, -\infty) \)

(d) \( (f / g)(x) = (\sqrt{x + 4}) / (x^2 - 2) \), and Bishop \( f / g = \{x \in \mathbb{R} : x + 4 \geq 0 \wedge x^2 - 2 \neq 0\} = [-4, -\sqrt{2}] \cup [-\sqrt{2}, \sqrt{2}] \cup [\sqrt{2}, +\infty) \)

(e) \( (g / f)(x) = ((2 - x^2) / (\sqrt{x + 4})) \), and Bishop \( g / f = \{x \in \mathbb{R} : x + 4 \geq 0 \wedge x + 4 \neq 0\} = ]-4, +\infty[ \)

Problem: Being \( f(x) = x - 2 \) and \( g(x) = x + 7 \), determine the functions and the field of composite function:

(a) \( (f \circ g)(x) = (x + 7) - 2 \), and Bishop \( f \circ g = \mathbb{R} \)

(b) \( (g \circ f)(x) = (x - 2) + 7 \) and Bishop \( g \circ f = \mathbb{R} \)

(c) \( (f \circ f)(x) = (x - 2) - 2 \), and Bishop \( f \circ f = \mathbb{R} \)

(d) \( (g \circ g)(x) = (x + 7) + 7 \), and Dom \( g \circ g = \mathbb{R} \)

Problem: Being \( f(x) = x^2 - 2 \) and \( g(x) = 1 / x \), determine the functions and the field of composite function:

(a) \( (f \circ g)(x) = (1 / x)^2 - 1 \) and Bishop \( f \circ g = \{x \in \mathbb{R} : x \neq 0\} = \{0\} \mathbb{R}^{-} \)

(b) \( (g \circ f)(x) = (1 / ((x^2 - 1))) \) and Dom \( g \circ f = \{x \in \mathbb{R} : x^2 - 1 \neq 0\} = ]-\infty, -1 \cup 1, 1\} \cup 1, +\infty[ \)
Problem: Being \( f(x) = \sqrt{x} \) and \( g(x) = \frac{1}{x} \), determine the functions and the field of composite function:

(a) \((f \circ g)(x) = \sqrt{\frac{1}{x}}\), and Bishop \((f \circ g) = \{x \in \mathbb{R}: \frac{1}{x} \neq 0\} = [-\infty, 0) \cup (0, +\infty)\]

[(b) \((g \circ f)(x) = \sqrt{\frac{1}{x}}\), and \(\text{Dom}(g \circ f) = \{x \in \mathbb{R}: x \geq 0\} = \mathbb{R}_-\]

[(c) \((f \circ f)(x) = \sqrt{\sqrt{x}} = \sqrt[4]{x}\), and Bishop \((f \circ f) = \{x \in \mathbb{R}: x \geq 0\} = [0, +\infty)\]

[(d) \((g \circ g)(x) = \frac{1}{\sqrt{\frac{1}{x}}} = x\), and \(\text{Dom}(g \circ g) = \{x \in \mathbb{R}: x \neq 0\} = \{0\}\]

Problem: Being \( f(x) = \sqrt{x^2-1} \) and \( g(x) = \sqrt{x-1} \), determine the functions and the domain of the resulting function:

(a) \((f \circ g)(x) = \sqrt{\left(\sqrt{x-1}\right)^2 - 1} = \sqrt{x-2}\), and Bishop \((f \circ g) = \{x \in \mathbb{R}: x-1 \geq 0 \wedge x-2 \geq 0\} = [2, +\infty)\]

[(b) \((g \circ f)(x) = \sqrt{\left(\sqrt{x^2-1}\right) - 1}\), and \(\text{Dom}(g \circ f) = \{x \in \mathbb{R}: x \geq 0 \wedge \sqrt{x^2-1} - 1 \geq 0\} = (-\infty, -\sqrt{2}] \cup [\sqrt{2}, +\infty)\]

[(c) \((f \circ f)(x) = \sqrt{\left(\sqrt{x^2-1}\right)^2 - 1} = \sqrt{x^2-2}\), and \(\text{Dom}(f \circ f) = \{x \in \mathbb{R}: x \geq 0 \wedge x \geq 0\} = (-\infty, -\sqrt{2}] \cup [\sqrt{2}, +\infty)\]

[(d) \((g \circ g)(x) = \sqrt{\left(\sqrt{x-1}\right)^2 - 1}\), and \(\text{Dom}(g \circ g) = \{x \in \mathbb{R}: x \geq 0 \wedge \sqrt{x-1} - 1 \geq 0\} = [2, +\infty)\]

Problem: Given \( f(t) = \frac{|t+3|-|t|-3}{t} \) expresses \( f(t) \) without bars absolute value, if \( t \) is in the range given:

(a) \( 0, +\infty \),

\[ |T + 3 + \text{At} \geq 3 \text{ e } t + (t + 3) \text{ At} < 3 \]

and \(|T| = (t+At \geq 0 \wedge t+At \leq 0)\]

If \( t \in [0, +\infty) \) then we can write: \( f(t) = (|l + t + l | - | l + t | - 3) / t = (|l + t + t | - 3) / t = 0 \)

(b) \([-3,0)\]

\[ |T| \leq -3 \text{ then we can write: } f(t) = (|l + t + l | - | l + t | - 3) / t = (|l + t + (-t) - t | - 3) / t = 2 \]

(c) \(-\infty, -3\)

\[ |T| \leq -3 \text{ then we can write: } f(t) = (|l + T + l | - | T + t | - 3) / t = (|l + t + (-t) - t | - 3) / t = - (6 / t) \]
Problem: Determine whether the given function is even, odd or neither.

(a) \( f(x) = 1 + 3x^2 \)
\[ 2x^4 \cdot f(-x) = 2(-x)^4 \cdot 3(-x)^2 + 1 = 2x^4 \cdot 3x^2 + 1 \] when the function is even.
(b) \( f(x) = x^7 \)
\[ 5x^3 \cdot f(-x) = 5(-x)^3 \cdot 7(-x) = -x \cdot (5x^3 \cdot 7x) \]
then the function is odd
(c) \( f(s) = s^2 + 2s + 2 \)
\[ f(-s) = (-s)^2 + 2(-s) + 2 = s^2 - 2s + 2 \]
or even or odd.
(d) \( g(x) = x^6 \cdot 1 \)
\[ g(-x) = (-x)^6 \cdot 1 = 1 \] when the function is even.
(e) \( h(t) = 5t^7 + 1 \)
\[ h(-t)^7 + 1 = 1-5t^7 \] or pair or odd.
(f) \( f(x) = |x| \)
\[ f(-x) = |-x| = |x| \]
then the function is even.
(g) \( f(y) = \frac{(y^3 - y)}{(1 + y^2)} \)
\[ f(y) = \frac{((-y)^3 - y)}{((-y)^2 + 1)} = -y \cdot \frac{(y^3 + y)}{(1 + y^2)} \] or pair or odd
(h) \( g(z) = \frac{((z-1)}{(z+1)} \)
\[ g((-z)-1) / ((-z+1)) = ((1 + z) / (z-1)) \]
or pair or odd (but note that \( g(z) = (1 / (g(z))) \) ie \( g(z) = 1 \))

Problem: There is a function that is odd and even? What
if there has to satisfy \( f(x) = f(-x) \land f(x) = -f(-x) \) then \( f(-x) = -f(-x) \equiv 2f(x) = 0 \equiv f(x) = 0 \)

Problem: Show that \( f = g = \frac{2x-3}{(x + 3)} / 2 \) are inverse functions
\( (f \circ g)(x) = 2(((x + 3) / 2)) \cdot 3 = x \) and Dom \( (f \circ g) = \mathbb{R} \)
\( (g \circ f)(x) = (((2x-3) + 3) / 2) = x \) and Dom \( (g \circ f) = \mathbb{R} \)

Logo are inverse functions.

Problem: Show that \( f = g = \sqrt{x^2} / x \) are inverse functions
\( (f \circ g)(x) = (\sqrt{x})^2 = x \) and Dom \( (f \circ g) = [0, + \infty) \)
\( (g \circ f)(x) = \sqrt{(x^2)} = x \) and Dom \( (g \circ f) = [0, + \infty) \)

[Logo Are inverse functions Issue:.]

Set the function \( f \circ g \) determine your domain

(a) \( f(x) = \cos x \) g \( (x) = x^2 \)
\( (f \circ g)(x) = \cos x^2 \) and Bishop \( (f \circ g) = \mathbb{R} \)
(c) \( f(x) = x \), \( g(x) = 2x \)

\((f \circ g)(x) = \csc 2x\) and \(\text{Bishop}(f \circ g) = \{x \in \mathbb{R}: \sin 2x \neq 0\} = \{x \in \mathbb{R}: x \neq (\pi n) / 2\}\)

44 - Problem: Set the function \(f\) and \(g\) determine your domain

(a) \( f(x) = \sin x, g(x) = (1/(2x)) \)

\((f \circ g)(x) = \sin (1/(2x)),\) and \(\text{Bishop}(f \circ g) = \{x \in \mathbb{R}: 2x \neq 0\} = \{0\}\)

(c) \( f(x) = \tan x, g(x) = x + \pi \)

\((f \circ g)(x) = \tan (x + \pi),\) and \(\text{Dom}(f \circ g) = \{x \in \mathbb{R}: \cos(x + \pi) \neq 0\} = \{x \in \mathbb{R}: x + \pi \neq (2n - 1)(\pi)\} = \{x \in \mathbb{R}: x + \pi \neq (2n - 3)(\pi)\}\)

### Activity 3 Sequences in R

#### Introduction

In this activity are presented sequences and subsequences, recursive sequences and some examples of notable sequences.

Define a convergent sequence, determines the limit of some threads and present some theorems on limits of sequences

#### Activity Details

**Sequences in R**

An ordered pair of real numbers is an ordered set of two real numbers: \((x, y)\)

an ordered triple of real numbers \((x, y, z)\)

a \(n\)-uplo ordered real numbers: \((x_1, x_2, ..., x_n)\)

a sequence (or series) of real numbers: \((x_1, x_2, ..., x_n, ...)\) is a \(N\) function to \(\mathbb{R}\)

\((x_n): N \to \mathbb{R}\)

a sequence is represented by \((x_n) = (x_1, x_2, x_3, ...)\), all of its terms by \(\{x_n\}\), and the general term for \(x_n\) where \(n \in \mathbb{N}\)

eg. the sequence given by \(x_1, \text{general term} \{n\} = (-1)^n\) can be written in full as an ordered set \((-1, 1, -1, 1, ...)\) while the terms of the sequence set is the set \([-1, 1]\).

A sequence may also be defined recursively (or inductively), indicating one or more initial elements \(x_1, x_2, ..., x_n\) and the formula for \(x_{n+1}\) in \(x_n\) function \(n, x_{n-1} \ldots x_{nk}\).

Ex. The sequence defined recursively \(x_1 = 5\) and \(n + x_1 \{1\} = ((x_n) / 2)\) can be written as \((x_n) = (5, (5/2), (5/4), (5/8), ...)\). The Fibonacci sequence, \(1 = x_1, x_2, 1\) and \(x_n \{n + 1\} = x_n \{n\} + x_n \{n-1\}\) can be written as \((1, 1, 2, 3, 5, 8, 13, ...)\)
Some notable sequences:

\[ x_ n = r, \text{ where } r \text{ is a constant (constant sequence)} \]

\[ x_ n = n \text{ (natural sequence)} \]

\[ x_ n = nr \text{ (arithmetic sequence first term and ratio } r) \]

\[ x_ n = r^n \text{ (geometric sequence of ratio } r) \]

\[ x_ n = \left(\frac{1}{n}\right) \text{ (sequence harmonic)} \]

\[ x_ n = \left(\frac{1}{n}\right)^p \text{ where } p > 0 \text{ (p-harmonic) sequence} \]

As sequences are N functions to R, satisfy the functions of algebra:

the sum of sequences (end to end), \( (x_ n) + (y_ n) = (x_ n + y_ n) \) is a binary operation in the set s (R) all sequences in R is associative, commutative, the neutral element is constant sequence \( x_ n = 0 \) and all sequences \( x_ n \) have symmetrical with -x_ n general term \( n \).

multiplication sequences (end to end), \( (x_ n) \cdot (y_ n) = (x_ n \cdot y_ n) \) operating binary in set s (R) all sequences in R is associative, commutative, the neutral element is constant sequence \( x_ n = 1 \) and only have reverse sequences in which no term is zero.

Convergence sequences R

A sequence converges in R if a LR number that satisfies the boundary condition:

\[ \forall \varepsilon > 0, \exists N (\varepsilon) \in \mathbb{N}: n > N \Rightarrow | x_ n - L | < \varepsilon \]

If so, the number L is called limit of the sequence and writes:

\[ \lim_{n \to \infty} x_ n = L \text{ or } x_ n \to L \]

natural number \( N (\varepsilon) \) is the order from which all of the following terms shall belong to vinhança ε L (can depend on ε):

\[ x_ n \in [L - \varepsilon, L + \varepsilon] \text{ for } n > N (\varepsilon) \]

If L is not the limit of a sequence then:

\[ \exists \varepsilon > 0: | x_ n - L | \geq \varepsilon, \forall n \in \mathbb{N} \]

ie, there is an neighborhood of L to which the following terms never belong

If no xR that is the limit of a sequence, this says if divergent:

\[ \forall x \in \mathbb{R}, \exists \varepsilon > 0: | x_ n - x | \geq \varepsilon, \forall n \in \mathbb{N} \]

Some theorems on limits of sequences in R:
The limit of a sequence, if any, is unique.

Any convergent sequence is bounded (the contrary proposition is not true, and the converse proposition also is not true).

Given the $x_\cdot$ sequences $\{n\} \to a; y_\cdot \{n\} \to b$, we have that:

\[ x_\cdot \{n\} + y_\cdot \{n\} \to a + b \]

\[ x_\cdot \{n\} \cdot A \cdot y_\cdot \{n\} \to a \cdot b \]

\[ x_\cdot \{n\} / y_\cdot \{n\} \to a / b \text{ only (} y_\cdot \{n\} \text{ is a sequence without zero terms) \}

\[ \text{if possibly } x_\cdot \{n\} \leq y_\cdot \{n\} \text{ then } \lim x_\cdot \{n\} \leq \lim y_\cdot \{n\} \]

\[ \text{If possibly } p \leq x_\cdot \{n\} \text{ then } \leq q \leq p \leq \lim x_\cdot \{n\} \leq q \]

\[ \text{If possibly } x_\cdot \{n\} \{n \leq z_\cdot \leq y_\cdot \{n\} \text{ then } a \leq \lim z_\cdot \{n\} \]

\[ \{\leq b \lim_\cdot n \to \infty} | x_\cdot \{n\} | = | a | \]

\[ \lim_\cdot \{n\} \to \infty \sqrt{(x_\cdot \{n\})} = \sqrt{a} \]

\[ \text{If } \lim (x_\cdot \{n + 1\}) / (x_\cdot \{n\}) < 1 \text{ then } \lim x_\cdot \{n\} = 0 \]

A sequence $(x_\cdot \{n\})$ growing $x_\cdot \{n\} \leq x_\cdot \{n + 1\}$ or descending $x_\cdot \{n \geq x_\cdot \}{n + 1}$ is said monotone.

A monotone sequence is convergent iff is limited (the monotone convergence theorem).

A subsequence $(x_\cdot \{k\})$ is a composite sequence $(x_\cdot \{n\})$ in $\mathbb{R}$ with a sequence $(n_\cdot \{k\})$ in $\mathbb{N}$ that preserves the order of the terms of the original sequence:

\[ x_\cdot \{k\} = x_\cdot \{n_\cdot \{k\} \}; n_\cdot \{k\} < n_\cdot \{j\} \Leftrightarrow k < j \]

Ex. $x_\cdot \{k\} = 1$ is a subsequence of $x_\cdot \{n\} = (-1)^n$ as making $n = 2k$ for $k \in \mathbb{N}$, the composition results in sequence

\[ x_\cdot \{k\} = x_\cdot \{n\} \cdot 2k \]

\[ = x_\cdot \{2k\} \]

\[ = (-1)^\cdot \{2k\} \]

\[ = ((-1)^\cdot \{k\} \]

\[ = 1 \cdot \{k\} \]

\[ = 1 \]

Theorems on subsequences:

If a sequence converges to a real number $L$ then any subsequence also converges to the same number $L$.

In particular, if $\lim_\cdot \{n \to \infty} x_\cdot \{n\} = \{G \lim_\cdot \{n \to \infty} x_\cdot \{n\} = L + k, \text{ for any } k \in \mathbb{N} \}

any sequence has a monotone subsequence (theorem of monotone subsequence).

Bolzano-Weierstrass theorem (version 1): Any bounded sequence in $\mathbb{R}$ has a convergent subsequence 2).
Bolzano-Weierstrass theorem (version) An infinite and limited subset of ER has an accumulation point.

A sequence is Cauchy (or fundamental) if
\[ \forall \varepsilon > 0, \exists N (\varepsilon) \in \mathbb{N}: m, n > N (\varepsilon) \Rightarrow |x_m - x_n| < \varepsilon \]

Theorems of Cauchy sequences

Any Cauchy sequence is limited. A sequence in R is convergent iff is Cauchy (Another way to say that R is complete)

A sequence contractive if there is a constant 0 < C < 1 such that
\[ |x_{n+2} - x_{n+1}| \leq C |x_{n+1} - x_n| \]

How can one diverge sequence? Or bounded but oscillating or is not defined.

If bounded but oscillating, has convergent subsequence. The limit of a convergent subsequence is called a sub-ceiling.

Since the sequence is bounded, the set of all sub is a non-empty and bounded subset of R logo has a tiny called lower limit, liminf \( x_\{n\} \), and a supreme called upper limit limsup \( x_\{n\} \)

If liminf \( x_\{n\} = \text{limsup } x_\{n\} \) then the sequence converges.

A \( E \subset \mathbb{R} \) subset is complete iff every Cauchy sequence in E converges.

Ex: R is complete.

A \( E \subset \mathbb{R} \) subset is (sequentially) compact iff any sequence in E has a convergent subsequence.

Ex: R is not compact, but any limited and closed subset of R is sequentially compact.

A \( E \subset \mathbb{R} \) subset is connected iff is an interval (open, closed or mixed)

Given a sequence \( \{x_n\} \) in R said that \( x_n \to +\infty \lim \) and writes \( x_n = +\infty \) to any \( \varepsilon > 0 \) exists an \( N (\varepsilon) \) such that if \( n \geq N (\varepsilon) \), then \( x_n > 1/\varepsilon \)

Non-defined sequences start to have a liminf \( x_\{n\} \) or \( \limsup \{n\} \) x_\{limsup\} in R (however, do not call these sequences convergent).

**Conclusion**

In this activity were presented sequences and subsequences, recursive sequences and some examples of notable sequences. Defined a convergent sequence, it was determined the limit of some threads and set out some theorems on limits of sequences.

**Evaluation**

Group a minimum of 5 and a maximum of 10 questions in each of the following evaluation test of 1 hour without consultation.

Choose different type questions in the preparation of each test, the changing parameters and database to prevent bonding.
Each question is worth between 10% and 20% depending on the number of selected questions.

Problem: Write the first terms of the sequence and try to intuit the limit:

\[
\left( \frac{2n^2 + 1}{3n^2 - n} \right) = \left( \frac{3}{2}, \frac{9}{10}, \frac{19}{24}, \frac{33}{44}, \ldots \right);
\]

Problem: Write the first terms of the sequence and try to intuit the limit:

\[
\left( \frac{2n^2 + 1}{3n^2} \right) = \left( \frac{2 + \left( \frac{1}{N^2} \right)}{3 \left( \frac{1}{n} \right)} \right) \rightarrow \left( \frac{3}{2} \right);
\]

Problem: Write the first terms of the sequence and try to intuit the limit:

\[
\left( \frac{3n^2 + 1}{2n^2 + n} \right) = \left( \frac{4}{3}, \frac{13}{10}, \frac{28}{21}, \frac{49}{36}, \ldots \right);
\]

Problem: Write the first terms of the sequence and try to intuit the limit:

\[
\left( \frac{3n^2}{2n^2 + n} \right) = \left( \frac{3 + \left( \frac{1}{N^2} \right)}{2 + \left( \frac{1}{n} \right)} \right) \rightarrow \left( \frac{3}{2} \right);
\]

Problem: Write the first terms of the sequence and try to intuit the limit:

\[
\left( \frac{E^n}{n} \right) = \left( e, \frac{e^2}{2}, \frac{e^3}{3}, \frac{e^4}{4}, \ldots \right); \text{ diverges (to justify later)}
\]

Problem: Write the first terms of the sequence and try to intuit the limit:

\[
\left( \frac{\log_{b} n}{n} \right) = \left( \frac{\log_{b} 2}{2}, \frac{\log_{b} 3}{3}, \frac{\log_{b} 4}{4}, \ldots \right); \text{ diverges (to justify later)}
\]

Problem: Write the first terms of the sequence and try to intuit the limit:

\[
\left( \frac{\sinh n}{\sin n} \right) = \left( 3966.1, 9886.3, 98870, -36059, \ldots \right); \text{ diverges}
\]

Problem: Write the first terms of the sequence and try to intuit the limit:

\[
\left( \sqrt{n + 1} - \sqrt{n} \right) = \left( 0.41421, 0.31784, 0.26795, 0.23607, \ldots \right); \text{ diverges}
\]

Problem: Write the first terms of the sequence and try to intuit the limit:

\[
\left( \frac{1 + \left( \frac{2}{n} \right)^n}{\left( \frac{1}{2} \right)^n} \right) = \left( 3, 4.46296.5, 0625, \ldots \right); \text{ diverges}
\]

Problem: Calculate the limit using the definition:

\[
| \left( \frac{1}{2n-1} \right) - 0 | < \epsilon \Rightarrow | \left( \frac{1}{2n-1} \right) | < \epsilon
\]

\[
\Rightarrow 2n-1 \geq (1 / \epsilon) \Rightarrow n > (1 / \epsilon)
\]
then \( N(\epsilon) = \max \{ 1, (1 / \epsilon) \} \) get the implication \( n > N(\epsilon) \Rightarrow | \left( \frac{1}{2n-1} \right) - 0 | < \epsilon \) (the end to the beginning chain implications)

Problem: Calculate the limit using the definition:

\[
| \left( \sqrt{n} \right) - 0 | < \epsilon \Rightarrow | \left( \sqrt{n} \right) | < \epsilon \Rightarrow n > (1 / \epsilon)
\]
N(ε) = (1 / (ε²)) then n > N(ε) ⇒ |1 / (1/n)| - 0| < ε (covering equivalences in reverse)

Problem: Calculate the limit using the definition:

\[ l ((5-n) / (3n + 2)) - (- (1/3)) | < ε \iff l ((3- (5-n) + 2 + 3n) / (3 (3n + 2))) | < ε \]

⇔ 0 < |((17) / (9n + 6))| < ε < 9n 6> ((17) / ε)

⇔ 9n> ((17) / ε) \iff n> ((17) / ε)

N(ε) = ((17) / ε) then n > N(ε) ⇒ l ((5-n) / (3n + 2)) - (- (1/3)) | < ε (covering the equivalences in reverse)

Problem: Calculate the limit: (1-(1-(1/n))^{a})/(1-(1-(1/n))^{b}))=((1-((1+((-1)/n))^{ⁿ})^{(a/n)})/(1-((1+((-1)/n))^{ⁿ})^{(b/n)}))→(a/b) (To justify later)

Problem: Prove the uniqueness of limits:

Suppose a_ {n} → L and a_ {n} → M. Taking ε = (1/2) | LM | there is an index p such that n > p ⇒

l a_ {n} - L | < (1/2) | LM | and also there is an index q such that n > q ⇒ l a_ {n} - M | < (1/2) | LM | .

Taking \( N = \max \{p, q\} \) has n > N ⇒ l a_ {n} - L | < (1/2) | LM | ∧ l a_ {n} - M | < (1/2) | LM | \iff

\( | a_ {n} - L | ≤ | a_ {n} - M | < (1/2) | LM | \)

\( L ≥ l a_ {n} - M | < | LM | and the triangular inequality is obtained that l L | ≤ l M | , a contradiction, then \( a_ {n} \) can not have two different limits.

Problem : Suppose a_ {n} → L . Consider the sequence ( | a_ {n} | ) . For any ε > 0 there exists N such that n > N⇒ | a_ {n} | ≤ | L | . The reciprocal is not true , for example | (-1) | n | → (1) | but (-1) | n does not converge . Note that to prove that \( | a_ {n} | = | a_ {n} - b | + | b | ≤ | a_ {n} - b | + | b | , do the same thing for | L | \)

\( | a_ {n} | = | a_ {n} - b | + | b | ≤ | a_ {n} - b | + | b | , do the same thing for | L | \)

Problem: ((2n -1) / (4n -1)) = (1/3) (3/7) (5 / (11)), ... ; Suppose ((2n -1) / (4n -1)) ≤ ((2 (n + 1) -1) / (4 (n + 1) -1))

⇔ (2n -1) (4n + 3) ≤ (2n + 1) (4n -1)

⇔ 8n² + 2n - 3 ≤ 8n² + 2n - 1

⇔ -3 ≤ -1

confirming that a_ {n} { ≤ a_ {n} + 1} for any n , then the sequence is increased .

Problem : (sin nπ ) = (0,0,0,...) is monotone ( increasing and decreasing if constant)

Problem: ((n³ -1) / n)) = (0, (7/2) (26) / 3), ... ; Suppose ((n³ -1) / n) ≤ (((n + 1)³ -1) / (n + 1))

⇔ (n³ -1) (n + 1) ≤ (n) (n + 1) 1 -³

⇔ n⁴ + n³ -n -1≤n⁴ 3n³ + + 3n²

⇔ -1≤2n³ 3n² + n +

confirming that a_ {n} { ≤ a_ {n} + 1} for any n , then the sequence is increased .
Problem: \( \frac{(2^n)}{(1+2^n)} = (\frac{2}{3}, \frac{4}{5}, \frac{9}{10}, \ldots) \); suppose \( \frac{(2^n)}{(1+2^n)} \leq \frac{(2^{n+1})}{(1+2^{n+1})} \)

\[ \Leftrightarrow \frac{2^n}{1+2^n} \leq \frac{2^{n+1}}{1+2^{n+1}} \]
\[ \Leftrightarrow 2 \times 2^n \geq 2 \times 2^{n+1} + 2^n \]
\[ \Leftrightarrow 0 \leq 2^n \]
confirming that \( a_n \leq a_{n+1} \) for any \( n \), then the sequence is increased.

Problem: \( \frac{(2n)!}{(5^n)} = (\frac{2}{5}, \frac{24}{25}, \frac{720}{125}, \ldots) \); Suppose \( \frac{(2n)!}{(5^n)} \leq \frac{(2(n+1))!}{(5^{n+1})} \)

\[ \Leftrightarrow 5^{n+1} \leq 5^n \frac{(2n)!}{(2n+1)!} \]
\[ \Leftrightarrow 5 \leq (2n+1)(2n+2) \]
confirming that \( a_n \leq a_{n+1} \) for any \( n \), then the sequence is increased.

Problem: \( \frac{n}{2^n} = (\frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \ldots) \); suppose \( \frac{n}{2^n} \geq \frac{n+1}{2^{n+1}} \)

\[ \Leftrightarrow 2n^n \geq n2^n + 2^n \]
\[ \Leftrightarrow 0 \geq (1-n)2^n \]
confirming that \( a_n \geq a_{n+1} \) for any \( n \), then the sequence is decreasing.

Problem: \( (1 \cdot 3 \cdot 5 \cdots (2n-1))/2^n = (\frac{1}{2}, \frac{3}{8}, \frac{15}{48}, \ldots) \); Suppose \( (1 \cdot 3 \cdot 5 \cdots (2n-1))/2^n \leq (1 \cdot 3 \cdot 5 \cdots (2n+1))/2^{n+1} \)

\[ 2n + 2 \leq n + 2n^2 \]
\[ 0 \leq n \cdot 2^{n^2} \]
confirming that \( a_n \geq a_{n+1} \) for any \( n \geq 1 \), then the sequence is decreasing for \( n > 1 \).

Problem: \( (3 - (1)^n = (3, 4, 2, 4, 2, \ldots) \); limited

Problem: \( (\frac{n}{3^n}) = (\frac{1}{9}, \frac{2}{27}, \frac{3}{81}, \ldots) \); suppose \( \frac{n}{3^n} \leq (\frac{n+1}{3^{n+1}}) \)

\[ n3^{n+2} \leq (n+1)3^{n+1} \]
\[ 0 \leq n - 3^{n+1} \]
confirming that \( a_n \geq a_{n+1} \) for any \( n \), then the sequence is decreasing. Furthermore the sequence is limited:

\[ 0 \leq n \leq 3^{n+1} \]
\[ 0 \leq (\frac{n}{3^{n+1}}) \leq 1 \]
then the sequence is convergent.
Problem: \( \left( \frac{2^n}{1 + 2^n} \right) \) had been verified that it is growing. As \\
\[ 0 \leq 2^n \leq 2^n + 1 \]
\[ 0 \leq \left( \frac{2^n}{2^n + 1} \right) \leq 1 \]
the sequence is also limited, so it is convergent.

Problem: \( \left( \frac{n}{2^n} \right) \) has been verified that it is decreasing. As it is also limited,
\[ 0 \leq n \leq 2^n \]
\[ 0 \leq \left( \frac{n}{2^n} \right) \leq 1 \]
Then the sequence is convergent.

Problem: \( \left( \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n n !} \right) \) had been verified that it is decreasing for \( n > 1 \). As it
\[ 0 \leq 1 \cdot 3 \cdot 5 \cdots (2n - 1) \leq 2^n n ! \]
\[ 0 \leq \left( \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n n !} \right) \leq 1 \]
Then the sequence is convergent.

\[ \text{NOTICE:} \]
\[ \left( \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n n !} \right) = \left( \frac{1}{2^n} \right) \prod_{k = 1}^{n} \left( \frac{2k - 1}{k} \right) \]
\[ \prod_{k = 1}^{n} \left( \frac{2k - 1}{2k} \right) < 1 \]
It appears that \( \lim_{n \to \infty} \left( \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n n !} \right) = 0 \)
\[ \left( \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n n !} \right) - 0 \leq \varepsilon \leq 0 < \left( \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n n !} \right) < \varepsilon \]
\[ \leq \left( \frac{2^n n !}{1 \cdot 3 \cdot 5 \cdots (2n - 1)} \right) \geq 1 / \varepsilon \]
\[ \varepsilon < \frac{n + 1}{\varepsilon^2} \leq n \geq \varepsilon^2 \]
then \( N(\varepsilon) = \max \{ 1, \log_2 (1 / \varepsilon) \} \) get the implication \( n > N(\varepsilon) \Rightarrow \left( \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n n !} \right) < \varepsilon \)

Problem: Prove by definition: \( \lim_{n \to \infty} \left( \frac{1}{\sqrt{n + 1}} \right) = 0 \)
\[ \left| \frac{1}{\sqrt{n + 1}} - 0 \right| \leq \varepsilon \leq 0 < (1 / \sqrt{n + 1}) < \varepsilon = 0 < (1 / (n + 1)) \leq \varepsilon^2 \]
\[ \leq 0 < n + 1 > \varepsilon^2 \leq n > \varepsilon^2 \]
\[ N(\varepsilon) = \text{then } \varepsilon^2 n > N(\varepsilon) \Rightarrow \left| \frac{1}{\sqrt{n + 1}} \right| - 0 \leq \varepsilon \]

Problem: Prove by definition: \( \lim_{n \to \infty} \left( n + 1 \right)^2 = + \infty \)

By definition \( \forall M \in \mathbb{R} \exists N \in \mathbb{N} : n > N \Rightarrow (n + 1)^2 > M \)

Be \( M \in \mathbb{R} \) an arbitrary number
\[ (n + 1)^2 > M \Rightarrow 0 < n + 1 > \sqrt{M} \Rightarrow n + 1 > \sqrt{M} \Rightarrow n > \sqrt{M} \]
Let $N = \sqrt{M}$ then $n > N \Rightarrow (n + 1)^2 > M$

**Problem:** Prove that: If $x_\{n\} \rightarrow y_\{x\} n \rightarrow y$ then $\forall \epsilon > 0 \exists N$ such that $n > N \Rightarrow |x_\{n\} - x| < (\epsilon / 2)$ and $n > M \Rightarrow |y_\{n\} - y| < (\epsilon / 2)$. So if $n > \max\{N, M\}$

\[
| (x_\{n\} - x) | + | (y - y_\{n\}) | < (\epsilon / 2) + (\epsilon / 2) = \epsilon
\]

Logo $(x_\{n\} - y_\{n\}) \rightarrow (xy)$

**Problem:** Prove that: If $x_\{n\} \rightarrow y_\{n\}$ then $\forall \epsilon > 0 \exists N$ such that $n > N \Rightarrow |x_\{n\} - x| < \epsilon$. So if $n > \max\{N, M\}$

\[
| (x_\{n\} - y_\{n\}) | < \epsilon
\]

Logo $(x_\{n\} - y_\{n\}) \rightarrow (x - y)$

In terms of the sequence, let $E_2$ be the half $E_1$ assembly or $[-M, 0] = \{0, M\}$ that has an infinite number of terms. It is divided in half for $E_2, E_3$, and so on. We obtain in this way a sequence of nested intervals:

$E_1 \supseteq E_2 \supseteq E_\ldots$

**33 - Problem:** Prove the theorem of Bolzano-Weierstrass any limited sequence $(a_\{n\})$ has convergent subsequence $(a_{n_\{k\}})$. Consider an arbitrary limited sequence $(a_\{n\})$. So $\exists M \forall n : |a_\{n\}| \leq M$ if $-M \leq a_\{n\} \leq M, \forall n$. Let $E_1 = [-M, M]$. The $E_1$ September has an infinite number

wherein the diameter of the next interval is always half the previous interval:

$DiamE_1 = 2M > DiamE_2 = (\frac{2M}{2}) > \ldots > DiamE_\{n\} = (\frac{2M}{2^{n-1}}) = (\frac{M}{2^{n-2}})$

As a consequence of this construction is obtained that

$\lim_{n \rightarrow \infty} DiamE_\{n\} = 0$

Choosing a term $x_\{n_\{k\}\}(k \epsilon E)$ each of the nested intervals is obtained sequence $(x_\{n\})$ a subsequence $(x_\{n_\{k\}\}) = (x_\{n_1\}, x_\{n_2\}, x_\{n_3\}, \ldots)$ which is Cauchy:

$\forall \epsilon > 0, \exists N : p, q > N \Rightarrow |x_\{n_\{p\}| - x_\{n_\{q\}| | < \epsilon$

because the intervals are embedded, if $|x_\{n_\{p\}| - x_\{n_\{q\}| | < \epsilon$ then there is $E_\{N\}$ such that $x_\{n_\{p\}|, x_\{n_\{q\}\| \epsilon E_\{N\}$ and $(C) DiamE_\{N\} = (\frac{M}{2^{n_\{p\}}}) < \epsilon$. Soon

$(\frac{M}{2^{n_\{p\} - 2}}) < \epsilon \Rightarrow (\frac{2^{n_\{p\} - 2}}{M}) > (\frac{1}{\epsilon})$
\[\Leftrightarrow 2^{n-2} > \frac{M}{\varepsilon}\]
\[\Leftrightarrow N > 2 + \log_2 \left(\frac{M}{\varepsilon}\right)\]

Making \(N(\varepsilon) = 2 + \lfloor \log_2 \left(\frac{M}{\varepsilon}\right) \rfloor\) it is concluded that
\[p, q > N(\varepsilon) \Rightarrow \{x_\{n\} \} \subseteq \mathbb{N} \Rightarrow \{x_\{n\} \} \subseteq \{x_\{n\} \} \implies |x_\{n\} - x_\{k\}| < \varepsilon\]

As \(R\) is complete, ie any Cauchy sequence converges, then the subsequence \(\{x_\{n\} \}\) also converges

**Activity 4 - Limits functions \(R\)**

**Introduction**

This activity is the calculation functions of limits on the finished line \(R\) making use of the symbols +\(\infty\),  -\(\infty\) to simplify the notation. Fifteen different types of limits are treated to a real function can have. The theorems about limits, after proven serve as a tool for problem solving. Indeterminate forms are clarified and raised in typical cases.

**Details of the activity**

Limits at infinity

The calculation of limits in \(R\) functions carried out on the finished line \(R\) .

The limit in a sequence of infinite can be generalized to an arbitrary function \(f: R \to R\) with domain unconfined above.

The actual number \(L \in R\) is the limit in the positive infinity of a function \(f\) iff
\[\forall \varepsilon > 0, \exists \ N \in \mathbb{N} : x > N \Rightarrow |f(x) - L| < \varepsilon\]

It is said that \(f(x) \to L\) as \(x \to +\infty\), and writes:
\[\lim_{x \to +\infty} f(x) = L\]

Similarly, the actual number \(L \in R\) is the limit in the negative infinity of a function \(f\) iff:
\[\forall \varepsilon > 0, \exists \ N \in \mathbb{N} : x < N \Rightarrow |f(x) - L| < \varepsilon\]

It is said then that \(f(x) \to L\) as \(x \to -\infty\), and writes:
\[\lim_{x \to -\infty} f(x) = L\]

When \(L\) is the limit at infinity ( positive or negative) of a function \(f\) then the line \(y = L\) is called a horizontal asymptote of \(f\).

Ex: If \(f(x) = \frac{1}{x}\) it appears that \(L = 0\) for any \(\varepsilon > 0\), if \(C = \frac{1}{\varepsilon}\), where \(x > N = \frac{1}{\varepsilon} \Rightarrow x > (1/\varepsilon) \Rightarrow (1/(|x|)) < \varepsilon\), obtain
\[|F(x) - 0| = \frac{1}{|x|} = \frac{1}{(|x|)} < \varepsilon\]

Ex: If \(L = 0\), for any \(\varepsilon > 0\), \(N = -(1/\varepsilon)\), where \(x < N = -(1/\varepsilon) \Rightarrow x'-(1/\varepsilon) \Rightarrow (1/(|x|)) < \varepsilon\), obtain
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\[ |F(x) - 0| = |1 / x| = (1 / |x|) < \varepsilon \]

The graph of the function \( f(x) = (1 / x) \) shows the horizontal asymptote \( y = 0 \)

Asymptotic values

The limits of a function at the edge of the area are called as asymptotic values.

Two functions \( f, g \) in \( \mathbb{R} \) are asymptotic or asymptotically equivalent (in the positive infinity) iff

\[ \lim_{x \to +\infty} (fg)(x) = 0 \]

An oblique asymptote of a function \( f(x) \) is a straight line \( y = mx + b \) asymptotically equal to \( f(x) \):

\[ \lim_{x \to +\infty} (f(x) - y) = 0 \]

The horizontal asymptote is a particular case of the oblique asymptote, where \( m = 0 \) then \( y = b \)

Example: To determine the oblique asymptote of \( f(x) = ((x^2 - 1) / (x + 1)) \) it is found that \( f(x) = (x - 2) + (3 / (x + 1)) \) just \( f(x) - (x - 2) = (3 / (x + 1)) \). As \( \lim_{x \to +\infty} (3 / (x + 1)) = 0 \) and also \( \lim_{x \to -\infty} (3 / (x + 1)) = 0 \), it follows that the oblique asymptote of \( f(x) \) it is the line \( y = x - 2 \).

Limits of a function in an area of accumulation point

In addition to the infinite limits, as in the case of threads, can be calculated limit of a function in either domain accumulation points.

An accumulation point the domain of \( f \) can be arbitrarily approached from two sides, if necessary:

the right side \( (to + (1 / n)) \to n \to \infty \) when the

the left side \( (a- (1 / n)) \to n \to \infty \) when the

Are defined lateral limits of a function in an area of accumulation point ::

\[ L \text{ is lateral limit the right of } f \text{ to satisfy the condition:} \]

\[ \forall \varepsilon > 0 \exists \delta > 0: 0 < x < \delta \Rightarrow |f(x) - L| < \varepsilon \]
it is written:
\[ \lim_{a^+} \{ x \to f(x) = L \} \]

L is lateral limit the right of f to satisfy the condition:
\[ \forall \varepsilon > 0, \exists \delta > 0, 0 < x < \delta \Rightarrow | f(x) - L | < \varepsilon \]

it is written:
\[ \lim_{a^+} \{ x \to f(x) = L \} \]

If the laterals limits of f(x) as x tends to the right to match the limit of f(x) as x tends to the left, then it will be referred to only as the limit of f(x) at the point a, and the first two conditions can be combined into one:

L is the limit of a function f at a point a \( \in \mathbb{R} \) iff there is a real number L \( \in \mathbb{R} \) satisfying the condition:
\[ \forall \varepsilon > 0 \exists \delta > 0: 0 < | x | < \delta \Rightarrow | f(x) - L | < \varepsilon \]

It is written:
\[ \lim_{X \to a} f(x) = L \]

Note that the limit of a function at a point a \( \in \mathbb{R} \) can exist even if f(a) does not exist.

infinite limits

It may be that there are laterals limits but the limit of a function at a point does not exist because they are not coincidental or simply because there is no real number with this property.

In the latter case it is said that the function has an infinite boundary or a vertical asymptote at the point a.

positive infinity limit:
\[ \lim_{a^+} \{ x \to f(x) = + \infty \} \]

negative infinity limit:
\[ \lim_{a^-} \{ x \to f(x) = - \infty \} \]

It is also considered the infinite lateral limits (positive or negative) to the left or right of a \( \in \mathbb{R} \) point:

\[ \lim_{a^+} \{ x \to f(x) = + \infty \} \]
\[ \lim_{a^-} \{ x \to f(x) = + \infty \} \]
\[ \lim_{a^+} \{ x \to f(x) = - \infty \} \]
\[ \lim_{a^-} \{ x \to f(x) = - \infty \} \]

Indeterminate Forms

The indefinite or indeterminate forms in the finished line \( \mathbb{R} \), take that name because they can take any value, depending on each case.
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For each particular case it is possible to calculate the value of “indeterminacy”, i.e. “to find the unknown.”

The technique of “lifting” of indeterminacies contains in essence the search function of the limits on the edge of your domain.

Indeterminacies group \((0/0)\) \((\pm \infty)/(\pm \infty)\) can be raised by emphasizing the highest degree term.

Indeterminacies the \(\infty-\infty\) group involving square roots can be solved by multiplying and dividing by the conjugate for indeterminacies group \((0/0)\).

Indeterminacies groups \(0^0\), \(1^{+\infty}\), \(0^{\pm\infty}\), \((\pm \infty)^0\) arising in the calculation limits the type functions \(f^{g}\), reduce to indeterminacies group \((0/0)\) \((\pm \infty)/(\pm \infty)\) by \(f^{g} = e^{\ln(f)^g} = e^{g\ln(f)}\)

For other general functions, Johann Bernoulli found a rule for calculating limits of fractions whose numerators and denominators tend to zero. The rule is now known as L'Hôpital's rule, in honor of the Marquis de l'Hôpital, who wrote the first introductory text of differential calculus (1696), where the rule first appeared. The rule of l'Hôpital will be studied later.

Theorems on limits

Theorems on limits are troubleshooting tools.

Considering real functions \(f, g\), numbers \(a, b, c, L_1, L_2, L, M, N \in \mathbb{R}\), occur, the following theorems:

- If \(\lim_{x \to a} f(x) = L_1\) and \(\lim_{x \to a} f(x) = L_2\) then \(L_1 = L_2\) (uniqueness limit)
- \(\lim_{x \to a} c = c\)
- \(\lim_{x \to a} x = a\)
- If \(\lim_{x \to a} f(x) = L\) and \(\lim_{x \to a} g(x) = M\) then \(\lim_{x \to a} (f + g) (x) = L + M\)
- If \(\lim_{x \to a} f(x) = L\) then \(\lim_{x \to a} (nf) (x) = nL\)
- If \(\lim_{x \to a} f(x) = L\) and \(\lim_{x \to a} g(x) = M\) then \(\lim_{x \to a} (f \cdot g) (x) = L \cdot M\)
- If \(\lim_{x \to a} f(x) = L\) then \(\lim_{x \to a} f^n (x) = L^n\)
- If \(\lim_{x \to a} f(x) = L\) and \(\lim_{x \to a} g(x) = M\) then \(\lim_{x \to a} \left( \frac{f}{g} \right) (x) = \frac{L}{M}\) to \(g(x) \neq 0\) and \(M \neq 0\) in vizinhnača of the
- If \(\lim_{x \to a} f(x) = L e^{(p/q)} \in \mathbb{Q}\) for \(p \neq 0\) then \(\lim_{x \to a} \{ f^{(p/q)} (x) \} = \{ U^{(p/q)}\}\)
- \(\lim_{x \to a} f(x) = x \to L \iff \lim_{x \to a} \{ f(x) - L \} = 0\) (vertical displacement)
- \(\lim_{x \to a} f(x) = L \iff \lim_{t \to 0} T f(t + a) = L\) (horizontal movement)
- If \(n \in \mathbb{N}\) then \(\lim_{x \to 0^+} (1/(x^n)) = +\infty\) and \(\lim_{x \to -\infty} (1/(x^n)) = -\infty\) if \(n\) is odd and \(+\infty\) if \(n\) is even
If \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = c \neq 0 \) then

\[
\lim_{x \to a} \left( \frac{g(x)}{f(x)} \right) = +\infty \text{ if } c > 0 \text{ and } f(x) \to 0 \text{ for positive values of } f(x)
\]

\[
\lim_{x \to a} \left( \frac{g(x)}{f(x)} \right) = -\infty \text{ if } c < 0 \text{ and } f(x) \to 0 \text{ for positive values of } f(x)
\]

\[
\lim_{x \to a} \left( \frac{g(x)}{f(x)} \right) = -\infty \text{ if } c < 0 \text{ and } f(x) \to 0 \text{ for negative values of } f(x)
\]

\[
\lim_{x \to a} \left( \frac{g(x)}{f(x)} \right) = +\infty \text{ if } c > 0 \text{ and } f(x) \to 0 \text{ for negative values of } f(x)
\]

The theorem remains valid if \( x \to a \) is replaced by \( x \to a^+ \) or \( x \to a^- \).

If \( \lim_{x \to a} f(x) = +\infty \) and \( \lim_{x \to a} g(x) = c \), then \( \lim_{x \to a} (f + g)(x) = +\infty \) (ditto \( x \to a^+ \) or \( a^- \)).

If \( \lim_{x \to a} f(x) = -\infty \) and \( \lim_{x \to a} g(x) = c \), then \( \lim_{x \to a} (f + g)(x) = -\infty \) (ditto \( x \to a^+ \) or \( a^- \)).

If \( \lim_{x \to a} f(x) = +\infty \) and \( \lim_{x \to a} g(x) = c \neq 0 \) then

\[
\lim_{x \to a} (f \cdot g)(x) = +\infty \text{ if } c > 0
\]

\[
\lim_{x \to a} (f \cdot g)(x) = -\infty \text{ if } c < 0
\]

The theorem remains valid if \( x \to a \) is replaced by \( x \to a^+ \) or \( x \to a^- \).

If \( \lim_{x \to a} f(x) = -\infty \) and \( \lim_{x \to a} g(x) = c \neq 0 \) then

\[
\lim_{x \to a}(f \cdot g)(x) = -\infty \text{ if } c > 0
\]

\[
\lim_{x \to a}(f \cdot g)(x) = +\infty \text{ if } c < 0
\]

The theorem remains valid if \( x \to a \) is replaced by \( x \to a^+ \) or \( x \to a^- \). If so \( n \in \mathbb{N} \lim_{x \to +\infty} \left( 1 / (x^n) \right) = 0 \) and \( \lim_{x \to -\infty} \left( 1 / (x^n) \right) = 0 \).

**Conclusion**

The calculation of limits of real functions \( R \) is done on the finished line \( R \) making use of the symbols \( +\infty, -\infty \) to simplify the notation. Like sequences, a function may have limits at infinity, i.e., when the variable \( x \) tends to positive infinity (or negative), \( x \in \mathbb{R} \pm \infty \). Contrary to \( \mathbb{N} \) that has no accumulation of points, a continuous variable can approach an accumulation point of the domain of \( f \) left or the right, resulting in the left lateral or right edge. The limit of a function at a point only exists if the lateral limits coincide. The threshold value can be one of the symbols \( +\infty, -\infty \), saying in this case that the limit is infinite. There are fifteen different kinds of limits that a real function may have and are summarized in tabular form in the body of the plug. The theorems about limits, after proven serve as a tool for problem solving. Indeterminate forms are clarified and raised a case.
**Evaluation**

Group a minimum of 1 and a maximum of 2 of the following questions in each evaluation test of 1 hour without consultation.

Choose different type questions in the preparation of each test, the changing parameters and database to prevent bonding.

Each question is worth between 50% and 100%, as the number of selected questions.

1 - Question: Prove (or disprove) by definition, the following limits: \( \lim_{x \to 1} (3x^2 - 7x + 2) = -2 \)

Answer:

\[
\forall \varepsilon > 0, \exists \delta > 0.0 < |x - 1| < \delta \iff |(3x^2, 2, 7x, +) - (-2)| < \varepsilon
\]

\[
|(3x^2, 2, 7x, +) - (-2)| < \varepsilon \iff 3|x| - 1 \iff x(4/3) | < \varepsilon
\]

\[
0 < |x - 1| < \delta \Rightarrow -1 < x, 1 < 1 \neq 1 \iff x
\]

\[
\Rightarrow 0 < x < 2 \iff x \neq 1
\]

\[
\Rightarrow - (4/3) < x - (4/3) < (2/3) \neq 1 \iff x
\]

\[
\Rightarrow |x - (4/3)| < (2/3) \neq 1 \iff x
\]

\[
3|x| - 1 \iff x - (4/3) | < \varepsilon
\]

\[
\iff |x - (4/3)| | < (\varepsilon/4) = \delta
\]

choosing \( \delta = \min \{1, (\varepsilon/4)\} \) the implication is rebuilt. Soon \( \varepsilon = 12.02 \) then \( \delta = \min \{1, (0.02/4)\} = 0.0005 \)

2 - Question: Prove (or disprove) by definition, the following limits: \( \lim_{x \to 5} (-4) = -4 \)

Answer:

\[
\forall \varepsilon > 0, \exists \delta > 0.0 < |x - 5| < \delta \iff |(-4)| - (-4)| = 0 < \varepsilon
\]

any \( \delta > 0 \), for example \( \delta = 1 \), meets the definition so any constant function is continuous.

3 - Question: Prove (or disprove) by definition, the following limits: \( \lim_{x \to 1 (4x + 3)} = 7 \)

Answer:

\[
\forall \varepsilon > 0, \exists \delta > 0.0 < |x - 1| < \delta \iff |(3 + 4x)| - 7 < \varepsilon
\]

\[
|(3 + 4x)| - 7 < \varepsilon \iff 4|x| - 4 < \varepsilon
\]

\[
\iff 4|x| < (\varepsilon/4) = \delta
\]
if \( \delta = (\varepsilon / 4) \) the proposition is true then 7 has the property required for the limit \( (4x + 3) \) as \( x \) tends to 1.

4 - Question: Prove (or disprove) by definition, the following limits: \( \lim_{x \to 3} (3x - 5) = 4 \)

Answer:
\[
\forall \varepsilon > 0, \exists \delta > 0.0 < |x - 3| < \delta \Rightarrow |(3x - 5) - 4| < \varepsilon
\]
\[
\L (3x - 5) - 4 | < E \Rightarrow |3x - 9| < \varepsilon
\]
\[
\L |3x - 9| < \varepsilon
\]
\[
\L |x - 3| < (\varepsilon / 3) = \delta
\]

if \( \delta = (\varepsilon / 3) \) the proposition is true just 4 has the property required for the limit \( (3x - 5) \) as \( x \) tends to 3.

5 - Question: Prove the following theorems on limits, considering real functions \( f \), \( g \) and real numbers \( a, b, c, L_1, L_2, L, M, N \in \mathbb{R} \), occur, the following theorems:

a) If \( \lim_{x \to a} f(x) = L_1 \) and \( \lim_{x \to a} f(x) = L_2 \) then \( L_1 = L_2 \) (uniqueness limit)

\[
\lim_{x \to a} (f + g)(x) = L + M
\]

For any \( \varepsilon > 0 \) let \( \delta_1 \) guarantee \( |f(x) - L_1| < \varepsilon / 2 \) and let \( \delta_2 \) guarantee \( |g(x) - M| < \varepsilon / 2 \). Then \( \delta = \min \{\delta_1, \delta_2\} \) ensures that

\[
|F(x) + g(x) - (L + M)| = |F(x) + g(x) - L - M| < \varepsilon / 2 + \varepsilon / 2 = \varepsilon
\]

b) \( \lim_{x \to a} x = a \)

For any \( \varepsilon > 0 \) let \( \delta \) guarantee \( |x - a| < \varepsilon \). Then this ensures that

\[
|nx - nL| = |n||x - a| < n\varepsilon
\]
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f) If the \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) then \( \lim_{x \to a} (f \cdot g)(x) = L \cdot M \)

Seeking the relationship between what you want to try and hypotheses:

\[
\left| (F \cdot g)(x) - LM \right| = \left| F(x)g(x) - Lg(x) + g(x) - LM \right|
\leq \left| g(x) \right| \left| f(x) - L \right| + \left| g(x) \right| \left| g(x) - M \right|
\]

Since \( g(x) \to M \), there \( \delta_1(g) > 0 \) ensures that \( \left| g(x) - M \right| < 1 \), using the absolute property value, \( \left| g(x) - M \right| < 1 \) is obtained that \( \left| g(x) \right| < 1 + \left| M \right| \), then

\[
\left| (F \cdot g)(x) - LM \right| < (1 + \left| M \right|) \left| f(x) - L \right| + \left| g(x) \right| \left| g(x) - M \right|
\]

For any \( \varepsilon > 0 \) let \( \delta_1 \) guarantee \( \left| f(x) - L \right| < \left( \varepsilon / (2 (1 + \left| M \right|)) \right) \) and let \( \delta_2 \) guarantee \( \left| g(x) - M \right| < \left( \varepsilon / (2 \left| L \right|) \right) \). Then \( \delta_1 = \min \left\{ \{ g \}, \delta_1, \delta_2 \right\} \) ensures that

\[
\left| (F \cdot g)(x) - LM \right| < (1 + \left| M \right|) \left| f(x) - L \right| + \left| g(x) \right| \left| g(x) - M \right|
\]

For any \( \varepsilon > 0 \) let \( \delta_1 \) guarantee \( \left| f(x) - L \right| < \left( \varepsilon / (2 (1 + \left| M \right|)) \right) \) and let \( \delta_2 \) guarantee \( \left| g(x) - M \right| < \left( \varepsilon / (2 \left| L \right|) \right) \). Then \( \delta_1 = \min \left\{ \{ g \}, \delta_1, \delta_2 \right\} \) ensures that

\[
\left| (F \cdot g)(x) - LM \right| < (1 + \left| M \right|) \left| f(x) - L \right| + \left| g(x) \right| \left| g(x) - M \right|
\]

h) \( \lim_{X \to a} f(x) = L \) and \( \lim_{X \to a} g(x) = M \) then \( \lim_{X \to a} \left( f / g \right)(x) = \frac{L}{M} \)

to \( g(x) \neq 0 \) and from a neighborhood \( M \neq 0 \)

Then \( g(x) \neq 0 \) and \( (1 / (G(x))) \) exists in a vicinity \( \delta_1 \{ g \} \) as \( M \neq 0 \), it follows that

\[
\left| (1 / (G(x))) - (1 / M) \right| = \left| ((Mg(x)) / (Mg(x))) - (1 / \left| g(x) \right|) \right| = \left| (1 / \left| g(x) \right|) - (1 / \left| M \right|) \right|
\]

You must now define \( g(x) \) before proceeding.

Since \( M \neq 0 \) then \( (\left| M \right| ) / 2) \) Ensures que the neighborhood \( V_\epsilon(\left| M \right| / 2) \) (M), defined by \( \left| g \right| < \left| M \right| / 2 \) is either all positive or all negative, depending on Whether \( M > 0 \) or if \( M < 0 \).

Since \( g(x) \to M \) is \( \varepsilon = (\left| M \right| / 2) \) \( \epsilon \) \( \left| \left| M \right| / 2 \right| > 0 \) Ensures que \( \left| g \right| < \left| M \right| < (\left| M \right| / 2). \)

Using the absolute value of property, are Obtained

\[
(\left| M \right| / 2) < g(x) < \left| M \right| < (\left| M \right| / 2)
\]

If \( M > 0 \), THEN \( F - (\left| M \right| / 2) = (\left| M \right| / 2) \) and \( F + (\left| M \right| / 2) = (3 \left| M \right| / 2) \) Then que are Obtained

\[
(\left| M \right| / 2) < g(x) < (3 \left| M \right| / 2)
\]

If \( M < 0 \), Then \( F + (\left| M \right| / 2) < 0 \), one can Then reverse the inequality and to obtain infor-
\[
0 < -M - ((| M |) / 2) < -g (x) < -M + ((| M |) / 2)
\]
\[
0 < | -M - ((| M |) / 2) | < | -g (x) | < | -M + ((| M |) / 2) |
\]
\[
0 < | -M | - ((| M |) / 2) | - | M | + ((| M |) / 2)
\]
\[
((|M|)/2)<|g(x)|<(3|M|)/2
\]
Results in any case, provided that \( M \neq 0 \), that
\[
( | G (x) -M | <( ( | M | ) / 2 ) ) \Rightarrow ( ( ( | M | ) / 2 ) < | g (x) | <( ( 3 | M | ) / 2 ) )
\]
and this implication is not reversible, the first inequality defined set is a second subset.

You can then go back to the equation ( <ref> inv </ref> ) with the delimitation of \( | g (x) | \) obtained above we obtain for the delimitation \( ( 1 / | g (x) | )\)
\[
(( | M | ) / 2 ) < | g (x) | <( ( 3 | M | ) / 2 ) \Rightarrow ( 2 / ( 3 | M | )) <( 1 / ( | g (x) | ) ) <( 2 / ( | M | ))
\]
Then for any \( \varepsilon > 0 \), there exists \( \delta _ { g } \) which guarantees the existence of \( ( 1 / ( G (x) ) ) \), there \( \delta _ { 1 } > 0 \) ensures that \( | g (x) | -M <( ( \varepsilon F / 2 ) / 2 ) \) and there is \( \delta _ { 2 } > 0 \) ensures that \( | g (x) | -M <( ( | M | ) / 2 ) \). Choosing \( \delta = \min \{ \delta _ { g }, \delta _ { 1 }, \delta _ { 2 } \} \) is guaranteed simultaneously these three conditions for that
\[
| ( 1 / ( G (x) ) ) - ( 1 / ( 1 / ( | F || g (x) | ) ) ) | Mg (x) | <( 2 / ( 1 | M | ) ) \Rightarrow ( | g (x) | <( ( | M | ) / 2 ) )
\]
Soon
\[
( 1 / ( G (x) ) ) \rightarrow ( 1 / ( 1 / M ) ) \text{ as } x \rightarrow a
\]
By the previous proposition,
\[
\lim_ {X \rightarrow a} ( (f / g) ) (x) = x \rightarrow \lim_ {a} (f (1 / (G (x)))) (x) = L (1 / M) = (L / M)
\]
i) If the \( \lim_ {x \rightarrow a} f (x) = L \) \( \epsilon (p / q) \in \mathbb{Q} \) for \( q \neq 0 \) then \( \lim_ {a} (f ^ {p/q}) (x) = L ^ {p/q} \)
Prove first the simplest proposition \( \lim_ {x \rightarrow a} f ^ {1/n} (x) = L ^ {1/n} \)
For it is recalled here the expression
\[
ab = (a ^ {1/n} - b ^ {1/n} ) \Sigma _ {k = 1} ^ {n} a ^ {((n-k+1)/n)} b ^ {((k-1)/n)}
\]
whose demonstration follows
\[
(A ^ {1/n} - b ^ {1/n} ) \Sigma _ {k = 1} ^ {n} a ^ {((n-k+1)/n)} b ^ {((k-1)/n)} = (a ^ {1/n} - b ^ {1/n} ) \Sigma _ {k = 1} ^ {n} a ^ {((n-k+1)/n)} b ^ {((k-1)/n)}
\]
\[
\Sigma _ {k = 1} ^ {n} a ^ {((n-k+1)/n)} b ^ {((k-1)/n)} = \Sigma _ {k = 1} ^ {n} a ^ {((n-k+1)/n)} b ^ {((k-1)/n)} - b ^ {1/n} \Sigma _ {k = 1} ^ {n} a ^ {((n-k+1)/n)} b ^ {((k-1)/n)}
\]
\[
= A + \Sigma _ {k = 1} ^ {n} a ^ {((n-k+1)/n)} b ^ {((k-1)/n)} - \Sigma _ {k = 1} ^ {n} a ^ {((n-k+1)/n)} b ^ {((k-1)/n)} - b ^ {1/n} \Sigma _ {k = 1} ^ {n} a ^ {((n-k+1)/n)} b ^ {((k-1)/n)}
\]
\[
= ab
\]
Note that when \( n \) is odd \( \text{Imf} \subset \mathbb{R} \) but when \( n \) is even \( \text{Imf} \subset \mathbb{R}_ \geq 0 \). So consider the case for any \( n \), then \( \text{Imf} \subset \mathbb{R}_ \geq 0 \) and \( L \geq 0 \)

\[
| f^{(1/n)}(x) - (L^{1/n}) | = | (f(x) - L) \sum_{k=1}^{n} f^{((nk)/n)}(x) (L^{((k-1)/n)}) |
\]

Since \( f(x) \to L \) in a neighborhood of a \( \delta_1 \) is possible to delimit \( | \sum_{k=1}^{n} f^{((nk)/n)}(x) (L^{((k-1)/n)}) | < K(U) \) and in the neighborhood \( \delta_2 \) is possible to delimit \( | (f(x) - L) | < \epsilon / (K(U)) \), then for \( \delta = \min(\delta_1, \delta_2) \) if

\[
\lim_{X \to a} f^{(1/n)}(x) = L^{(1/n)}
\]

To prove that \( \lim_{X \to a} f(x) = L \iff \lim_{t \to 0} T f(t + a) = L \) (horizontal displacement of origin or variable change)

Making the change of variable \( t = x \) has that \( \lim_{x \to a} t = \lim_{x \to a} (x) = 0 \), ie \( T \to 0 \) as \( x \to a \)

Reciprocamente if \( x = t + a \) to have that \( \lim_{t \to 0} x = \lim_{t \to 0} (t + a) = a \), ie \( x \to a \) when \( T \to 0 \)

So,

\[
\lim_{X \to a} f(x) = L
\]

Conversely ,

\[
\lim_{r \to 0} x \to f(x) = \lim_{t \to 0} T f(t + a) = U
\]

\[ i) \text{ If } n \in \mathbb{N} \text{ then } \lim_{x \to 0^+} (1/(x^n)) = + \infty \text{ and } \lim_{x \to 0^-} (1/(x^n)) = - \infty \text{ if } n \text{ is odd} \]

\[ + \infty \text{ if } n \text{ is even} \]

As \( 0 < x < (1/M) \equiv (1/x) > M > 0 \), choosing \( \delta = (1/M) \) we have that

\[ \forall M > 0.0 < x < \delta = (1/M) \Rightarrow (1/x) > M \]

soon

\[ \lim_{x \to 0^+} (x)(1/x) = + \infty \]

For the above proposition , and operating in the finished line \( R \), it follows that

\[ \lim_{x \to 0^+} (x)(1/x^n) = \{x \to 0^+ \lim_{x \to 0} (1/x)\} = n (+ \infty) = + \infty^n \]
For the case \( \lim_{x \to 0^-} x \left( \frac{1}{x^n} \right) \) as \( M < x < 0 \) it follows that \( 0 < -x < -M \iff \left( \frac{1}{-M} \right) < \left( \frac{1}{-x} \right) \), choosing \( \delta = \left( \frac{1}{-M} \right) \) we have that

\[
\forall M < 0, -\left( \frac{1}{-F} \right) = \frac{1}{-M} < \frac{1}{-x} \iff \frac{1}{M} < x \iff M < \frac{1}{x}
\]

soon

\[
\lim_{x \to 0^-} \left( \frac{1}{x} \right) = -\infty
\]

For the above proposition, and operating in the finished line \( \mathbb{R} \), it follows that

\[
\lim_{x \to 0^-} \left( \frac{1}{x^n} \right) = \left( \lim_{x \to 0^-} \left( \frac{1}{x} \right) \right)^n = (-\infty) = \left\{ \begin{array}{ll}
\infty & \text{if } n \text{ is odd} \\
+\infty & \text{if } n \text{ is even}
\end{array} \right.
\]

**Summary**

In this unit \( \mathbb{R} \) was characterized as a complete ordered field with the property of the Supreme. Any other set with the same properties is isomorphic to \( \mathbb{R} \).

The finished line was defined, with the introduction of symbols \(-, +\), and operations and exceptions to these symbols.

We present some concepts topological \( \mathbb{R} \) relative to notable points and subsets of \( \mathbb{R} \).

We introduced several basic functions for its analytical expression in the independent variable, including polynomial, rational, with radical, logarithmic, exponential and logarithmic.

It was shown how to solve equations or inequalities that can arise when calculating the function domain.

Sequences and subsequences were presented, recursive sequences and some examples of notable sequences.

Defined a convergent sequence, it was determined the limit of some threads and set out some theorems on limits of sequences.

It was introduced the concept of continuous function and presented the calculation functions of limits on the finished line \( \mathbb{R} \) making use of the symbols \(+\infty\), \(-\infty\) to simplify the notation.

Fifteen different types of limits have been presented a real function can have.

Finally we were presented some theorems on limits of real functions.
Unit Evaluation

Instructions

Group a minimum of 5 and a maximum of 10 questions in each of the following evaluation test of 1 hour without consultation.

Choose different type questions in the preparation of each test, the changing parameters and database to prevent bonding.

Rating criteria

Each question is worth between 10% and 20% depending on the number of selected questions.

Evaluation

solved exercises on functions of limits:

**Question 1.**

\[
\lim_{x \to 1} \left( \frac{\sqrt[3]{x} - 1}{x - 1} \right) = \frac{x - 1}{x - 1} \cdot \frac{x + 1}{x + 1} = \frac{1}{3}
\]

**Question 2.**

\[
\lim_{x \to 0} \left( \frac{x^2}{\ln \cos x} \right) = \lim_{x \to 0} \left( \frac{2}{-\tan x} \right) = -2
\]

\[
\lim_{x \to 0} \left( \frac{x^2}{\ln \cos x} \right) = \lim_{x \to 0} \left( \frac{\cos x + 1}{\cos x + 1} \right) = -1 \cdot \frac{x}{\sin x} \cdot \frac{x}{\sin x} \cdot \cos x + 1 = -2
\]

**Question 3.**

\[
\lim_{x \to 0} \left( \frac{e^x - e^{-x} - 2x}{\sin x} \right) = \lim_{x \to 0} \left( \frac{e^x + e^{-x} - 2}{1 - \cos x} \right) = 2
\]
Question 4.

\[
\lim_{x \to 0^+} (\tan x) = \ln x \quad \lim_{x \to 0^+} \left(\frac{x}{\ln x}\right)
\]

\[
\lim_{x \to 0^+} \left(\frac{1}{\ln x - x}\right) = 0
\]

Question 5.

\[
\lim_{x \to 0} \left(\frac{\sin x \cos x}{x - \sin x}\right) = x \lim_{x \to 0} \left(\frac{\cos x - \cos x \sin x}{1 - \cos x}\right)
\]

\[
\lim_{x \to 0} \left(\frac{x \sin x}{1 - \cos x}\right) = \lim_{x \to 0} \left(\frac{(x \sin x) + (1 - \cos x)}{x + x \ln x - 1}\right) = 2
\]

Question 6.

\[
\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x - 1}\right) = \lim_{x \to 1^+} \left(\frac{1 - x - \ln x}{x - 1 - x \ln x}\right)
\]

\[
\lim_{x \to 1^+} \left(\frac{x - 1}{x + x \ln x - 1}\right) = \lim_{x \to 1^+} \left(\frac{x - 1}{x(1 + x)}\right) = \frac{1}{2}
\]

Question 7.

\[
\lim_{x \to 0} \left(\frac{\tan x - \sin x}{x^3}\right) = \lim_{x \to 0} \left(\frac{\tan x (1 - \cos x)}{x^3}\right)
\]

\[
\lim_{x \to 0} \left(\frac{1 - \cos x}{x^3}\right) = \frac{1}{2}
\]

Question 8.

\[
\lim_{x \to (\pi/2)^-} (2 \tan x \cdot \sec x) = \lim_{x \to (\pi/2)^-} (2x \cdot (\sin x) / (\cos x) - \pi (1 / (\cos x)))
\]

\[
\lim_{x \to (\pi/2)^-} \left(\frac{2 \sin x - \pi x}{(2 \sin x - \pi)^2 / (\cos x)\right)} = -2
\]
Question 9.
\[
\lim_{x \to 0^+} \left( \frac{1}{1-x} - \frac{2}{x^2-1} \right) = 1
\]

Question 10.
\[
\lim_{x \to 1} \left( \frac{1}{1-x} - \frac{2}{x^2-1} \right) = \frac{x + 1 - 2}{x^2-1} = \frac{(x-1)}{(x-1)(x + 1)} = \frac{1}{2}
\]

Question 11.
\[
\lim_{x \to 0^+} \left( \frac{e^{-\frac{1}{x}}}{x} \right) = \lim_{x \to 0^+} \left( \frac{1}{x} \right) \frac{1}{e^{\frac{1}{x}}} = 0
\]

Question 12.
\[
\lim_{x \to 0} \left( \frac{\sin x \, x \cos x}{x^2 \sin x} \right) = \frac{1}{3}
\]

Question 13.
\[
\lim_{x \to 0^+} \left( \frac{\arctan x - x}{x^3} \right) = \frac{1}{3}
\]

Question 14.
\[
\lim_{x \to \infty} (1 + \frac{1}{x})^x = e \quad \text{and} \quad \lim_{u \to 0} \left( \frac{1}{u} \ln (1-u) \right) = -1
\]
\[
\lim_{x \to \infty} (1 + (\alpha / x))^x = \{u \lim_{u \to 0} \ln (1 + \alpha u)^{(1 / u)}\} = E^{\alpha}
\]

**Question 15.**

\[
\lim_{x \to +\infty} \left(\frac{1}{x} + \frac{\ln (x + 1)}{x^2}\right) = \lim_{x \to \infty} \frac{(x + \ln (x + 1))}{x^2}
\]

\[
\lim_{x \to -\infty} \left(\frac{1}{x} + \frac{\ln (|x + 1|)}{x^2}\right) = 0
\]

**Question 16.**

\[
\lim_{x \to \infty} \left(1 - \sin \left(\frac{1}{x}\right)\right)^{x^2} = \lim_{x \to \infty} \left(1 - \frac{2 - \sin \left(\frac{1}{x}\right)}{2}\right)^{2x} = e^{-2}
\]

**Question 17.**

\[
\lim_{x \to \infty} x^2e^{-x} = \{x \lim_{x \to \infty} \ln e^{x^2}\} = \lim_{x \to \infty} \frac{(2)}{(e^x)} = 0
\]

**Question 18.**

\[
\lim_{x \to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = x \to 0^+ \lim_{x} \left(\frac{1}{x} - \frac{1}{x}\right) = 0
\]
Question 19.

\[ \lim_{x \to 2} \frac{(1 / (\ln (x-1))) - (1 / (x-2))}{(2-x) - \ln (1-x)} = \lim_{x \to 2} \frac{((x-2) - \ln (1-x))}{((x-2) + (x-2))} \]

\[ \lim_{x \to 2} \frac{1}{(x-1) + 1} = \frac{1}{2} \]

Question 20.

\[ \lim_{x \to 0} \frac{\tan x}{x - \sin x} = \lim_{x \to 0} \frac{\sec^2 x - 1}{1 - \cos x} \]

\[ \lim_{x \to 0} \frac{2 \sec x (\tan x \sec x)}{\sin x} = 2 \]

Question 21.

\[ \lim_{x \to 0^+} (1 + 2x)^{1/x} = \lim_{x \to 0^+} (1 + \frac{2}{1/x})^{1/x} = e^{-2} \]

Question 22.

\[ \lim_{x \to 0^+} \frac{1 - \sin 2x}{x} = +\infty \]

Question 23.

\[ \lim_{x \to \infty} \ln x = \lim_{x \to +\infty} (\ln x) / (e^{x}) = 0 \]

Question 24.

\[ \lim_{x \to 0^+} (1 - 2x)^{1/x} = \lim_{x \to 0^+} (1 - \frac{2}{1/x})^{1/x} = e^{-2} \]
Question 25.

\[ \lim_{x \to 0^-} \left( \frac{\cos(\frac{\pi}{2} - x)}{\arctan x} \right) = \lim_{x \to 0^-} \left( \frac{\sin(\frac{\pi}{2} - x)}{\frac{1}{1 + x^2}} \right) \]

\[ \lim_{x \to 0^-} = 1 \]

Question 26.

\[ \lim_{x \to 0^+} (1 - 2x)^{\cot x} = \lim_{x \to 0^-} (1 + \frac{-2}{1/x})^{1/x} = e^{-2} \]

Question 27.

\[ \lim_{x \to \frac{\pi}{2}^-} (\sec x + 5) / (3\tan x) = \lim_{x \to \frac{\pi}{2}^-} (\tan x\sec x) / (3\sec^2 x) \]

\[ \lim_{x \to \frac{\pi}{2}^-} = \frac{1}{3} \]

Question 28.

\[ \lim_{x \to \infty} x \ln (1 + \frac{a}{x}) = \lim_{x \to \infty} \ln (1 + \frac{a}{x})^x \]

\[ \ln \lim_{x \to \infty} = e^a = a \]

Question 29.

\[ \lim_{x \to +\infty} x - \sqrt{x^2 - 10} = \lim_{x \to +\infty} x - \sqrt{1 - \frac{10}{x^2}} \]

\[ = \lim_{x \to +\infty} x \left(1 - \sqrt{1 - \frac{10}{x^2}}\right) / (1 + \sqrt{1 - \frac{10}{x^2}}) / (1 + \sqrt{1 - \frac{10}{x^2}}) \]

\[ \lim_{x \to +\infty} x \left(1 - \sqrt{1 - \frac{10}{x^2}}\right) / (1 + \sqrt{1 - \frac{10}{x^2}}) / (1 + \sqrt{1 - \frac{10}{x^2}}) = 0 \]

Question 30.

\[ \lim_{x \to +\infty} x - \sqrt{x^2 - 10x} = \lim_{x \to +\infty} x - \sqrt{1 - \frac{10}{x}} \]

\[ = \lim_{x \to +\infty} x \left(1 - \sqrt{1 - \frac{10}{x}}\right) / (1 + \sqrt{1 - \frac{10}{x}}) / (1 + \sqrt{1 - \frac{10}{x}}) \]

\[ \lim_{x \to +\infty} x \left(1 - \sqrt{1 - \frac{10}{x}}\right) / (1 + \sqrt{1 - \frac{10}{x}}) / (1 + \sqrt{1 - \frac{10}{x}}) = 5 \]
Question 31.
\[
\lim_{x \to 0^+} (e^x + x)^{\left(\frac{1}{x}\right)} = e^{\lim_{x \to 0^+} \left(\frac{\ln(\lim_{x \to 0^+} (e^x + x))}{x}\right)}
\]
\[
= e^{\lim_{x \to 0^+} \left(\frac{\ln(e^x + x) + 1}{\ln(e^x + x)}\right)}
\]
\[
= e^2
\]

Question 32.
\[
\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right) = \left(\lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}\right)
\]
\[
= 1
\]

Question 33.
\[
\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x \to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)
\]
\[
\lim_{x \to \frac{\pi}{2}} \left(\frac{1 - \sin x}{\cos x}\right)
\]
\[
\lim_{x \to \frac{\pi}{2}} \left(\frac{-\cos x}{-\sin x}\right)
\]
\[
= 0
\]

Question 34.
\[
\lim_{x \to 0^+} (x^x)^x = \lim_{x \to 0^+} (x^x)^{\left(\frac{\ln x}{x^2}\right)}
\]
\[
= e^{\lim_{x \to 0^+} \left(\frac{\ln x}{x^2}\right)}
\]
\[
= e^{\lim_{x \to 0^+} \frac{1}{-2x^4}}
\]
\[
= e^{\lim_{x \to 0^+} \frac{x^4}{-2}}
\]
\[
= 1
\]

Question 35.
\[
\lim_{x \to 0} \frac{(e^x - e^{-x})}{\sin x} = \lim_{x \to 0} \frac{(e^x + e^{-x})}{\cos x}
\]
\[
= 2
\]

Question 36.
\[
\lim_{x \to 0^+} \left(\frac{1}{x} + \ln x\right) = \lim_{u \to +\infty} + (u + \ln(1/u))
\]
\[
\lim_{u \to +\infty} (e^u)^{\ln(1/u)}
\]
\[
\lim_{u \to +\infty} (\frac{e^u}{u})
\]
\[
\lim_{u \to +\infty} \frac{(e^u)}{1}
\]
\[
= +\infty
\]
Question 37.
\[
\lim_{x \to +\infty} (1 + \frac{1}{3x}) = x \lim_{x \to +\infty} \ln (1 + \frac{1}{3x})
\]
\[
= E^{\lim_{x \to +\infty} \left(\ln (1 + \frac{1}{3x})\right) / \left(\frac{1}{x}\right)}
\]
\[
= E^{\lim_{x \to +\infty} \left(\frac{-\frac{1}{3x^2}}{\frac{1}{x^2}} \right)}
\]
\[
= E^{\lim_{x \to +\infty} \left(\frac{1}{3} \right)}
\]
\[
\lim_{x \to +\infty} (1 + \frac{1}{3x}) = x \lim_{x \to +\infty} (1 + \frac{1}{3x}) = E^{\lim_{x \to +\infty} \left(\frac{1}{3}\right)}
\]

Question 38.
\[
\lim_{x \to 0} (1 - 2x)^{\frac{2}{x}} = \lim_{x \to 0} \left(1 + \frac{-2}{1} / \left(\frac{1}{x}\right)\right)^{\frac{1}{x}}
\]
\[
= e^{-4}
\]

Question 39.
\[
\lim_{x \to 0^+} \left(\frac{E^{2x}}{x \cos x}\right) = \left(\frac{1}{0^+}\right)
\]
\[
= +\infty
\]

Question 40.
\[
\lim_{x \to 0} \left(\frac{x - \sin x}{x - \tan x}\right) = \lim_{x \to 0} \left(\frac{1 - \cos x}{1 - \sec^2 x}\right)
\]
\[
\lim_{x \to 0} \left(\frac{-\sin x}{2 \sin x \sec^3 x}\right)
\]
\[
= \frac{-1}{2}
\]

Question 41.
\[
\lim_{x \to 0} x (1 + \tan x)^{\frac{1}{x}} = \lim_{x \to 0} x \ln (\tan 1 + x)^{\frac{1}{x}}
\]
\[
\text{And } \lim_{x \to 0} \left(\frac{\ln (1 + \tan x)}{x}\right)
\]
\[
\text{And } \lim_{x \to 0} \left(\frac{\sec^2 x}{(1 + \tan x)}\right)
\]
\[
= e
\]

Question 42.
\[
\lim_{x \to 1} (\frac{1}{\ln x} - \frac{x}{\ln x}) = \lim_{x \to 1} (\frac{1 - x}{\ln x})
\]
\[
\lim_{x \to 1} = (\frac{1}{x}) - (\frac{1}{x})
\]
\[
= -1
\]
Question 43.

\[ \lim_{x \to 0^+} \left( \frac{2}{\sin^2 x} - \frac{1}{1 - \cos x} \right) = \lim_{x \to 0^+} \left( \frac{2}{1 - \cos^2 x} - \frac{1}{1 - \cos x} \right) \]

\[ = \lim_{x \to 0^+} \left( \frac{1 - \cos x}{1 - \cos^2 x} \right) \]

\[ = \lim_{x \to 0^+} \left( \frac{1 - \cos x}{1 - \cos^2 x} \right) \]

\[ = \frac{1}{2} \]

Question 44.

\[ \lim_{x \to \infty} \left( 1 - \frac{3}{x} \right) = x^{-3} \]

Question 45.

\[ \lim_{x \to \infty} \left( \frac{x^3}{e^x} \right) = \lim_{x \to \infty} \left( \frac{3x^2}{e^x} \right) \]

\[ = \lim_{x \to \infty} \left( \frac{6x}{e^x} \right) \]

\[ = 0 \]

Question 46.

\[ \lim_{x \to 0} \left( 1 + \sin x \right)^{\frac{1}{3x}} = \lim_{z \to 0^+} \left( \left( 1 + x \right)^{1/x} \right)^{\frac{1}{3}} \]

\[ = e^{\frac{1}{3}} \]

Question 47.

\[ \lim_{x \to 0} \left( \frac{x^2}{4\tan x} \right) = \lim_{x \to 0} \left( \frac{x^2}{\tan x} \right) \frac{e^{2x}}{4} \]

\[ = \frac{1}{4} \]

Question 48.

\[ \lim_{x \to 0^+} \left( x - \cot \left( \frac{1}{x} \right) \right) = \lim_{x \to 0^+} \left( x\cot 1 - x \right) \]

\[ = 0 \]
Question 49.
\[
\lim_{x \to 0^+} ((1/x)^{\sin x}) = e^{\lim_{x \to 0^+} \ln ((1/x)^{\sin x})}
\]
\[
= e^{\lim_{x \to 0^+} -\sin x \cdot \ln x}
\]
\[
= e^{\lim_{x \to 0^+} - \left( \frac{\ln x}{x \cdot \csc x} \right)}
\]
\[
= e^{\lim_{x \to 0^+} - \frac{\tan x \cdot \sin x}{x}}
\]
\[
= 1
\]

Question 50.
\[
\lim_{x \to \infty} (\frac{2+x}{x})^{x} = e^{\lim_{x \to \infty} \ln \left(\frac{2+x}{x}\right)^{x}}
\]
\[
= e^{\lim_{x \to \infty} -x}
\]
\[
= e^{\lim_{x \to \infty} -\frac{1}{2}}
\]
\[
= e^{0}
\]

Question 51.
\[
\lim_{x \to 0^+} (\frac{1}{x} - (\frac{1}{e^{x}}} \cdot \frac{x}{x}) = \lim_{x \to 0^+} \frac{(e^{-1} \cdot x - x)}{x \cdot (e^{-1} \cdot x + x^x)}
\]
\[
= \lim_{x \to 0^+} \frac{e^{-1} \cdot x}{e^{-1} \cdot x + x^x}
\]
\[
= \lim_{x \to 0^+} \frac{e^{-1} \cdot x}{2e^{-1} \cdot x + x^x} = \frac{1}{2}
\]

Pergunta 52.
\[
\lim_{x \to 1^+} (\frac{1}{\ln x} + (\frac{1}{x-1})) = \lim_{x \to 1^+} \frac{(\ln x + x-1)}{(x-1) \ln x}
\]
\[
= \lim_{x \to 1^+} \frac{1}{-\sin x}
\]
\[
= \infty
\]

Question 53.
\[
\lim_{x \to \pi} (3-(\frac{x}{\pi}))^{\csc x} = e^{\lim_{x \to \pi} \frac{\ln (3-(\frac{x}{\pi}))}{\sin x}}
\]
\[
= e^{\lim_{x \to \pi} \frac{1}{-1}} = e^{-1}
\]

Question 54.
\[
\lim_{x \to -(\pi/2)} ((\frac{\pi}{2})+x) \cdot \sec x = \lim_{x \to -(\pi/2)} \frac{((\frac{\pi}{2})+x)}{\cos x}
\]
\[
= \lim_{x \to -(\pi/2)} \frac{1}{-\sin x}
\]
\[
= 1
\]
Question 55.
\[
\lim_{x \to 1} \frac{1-x+\ln x}{1+\cos \pi x} = \lim_{x \to 1} \frac{-1+\frac{1}{x}}{-\pi \sin \pi x}
\]
\[
= \lim_{x \to 1} \frac{x-1}{\pi x \sin \pi x}
\]
\[
= \lim_{x \to 1} \frac{1}{\pi \sin \pi x + \pi^2 x \cos \pi x}
\]
\[
= -\frac{1}{\pi^2}
\]

Question 56.
\[
\lim_{x \to 0} (1+3x)^{\csc x} = e^{\lim_{x \to 0} \frac{\ln(1+3x)}{\sin x}}
\]
\[
= e^{\lim_{x \to 0} \frac{3}{(1+3x)\cos x}}
\]
\[
= e^3
\]

Question 57.
\[
\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \frac{1-\sin x}{\cos x} \cdot \frac{1+\sin x}{1+\sin x}
\]
\[
= \lim_{x \to (\pi/2)^-} \frac{1-\sin^2 x}{(1+\sin x)\cos x}
\]
\[
= \lim_{x \to (\pi/2)^-} \frac{\cos x}{(1+\sin x)}
\]
\[
= 0
\]

Question 58.
\[
\lim_{x \to (\pi/2)^-} (\cos x) \ln \cos x = \lim_{x \to (\pi/2)^-} ((\ln \cos x)/(1/(\cos x)))
\]
\[
= \lim_{x \to (\pi/2)^-} ((-\sin x)/(\cos x)) \cdot (-1/(\cos^2 x \sin x))
\]
\[
= \lim_{x \to (\pi/2)^-} \cos x \sin^2 x = 0
\]

Question 59.
\[
\lim_{x \to \infty} (1+8x^2)^{(1/x^2)} = e^{\lim_{x \to \infty} \frac{\ln(1+8x^2)}{x^2}}
\]
\[
= e^{\lim_{x \to \infty} (16x)/(2x(1+8x^2))}
\]
\[
= e^{\lim_{x \to \infty} (8/(1+8x^2))}
\]
\[
= 1
\]

Pergunta 60.
\[
\lim_{x \to 0} \frac{1}{\ln(1+x)} - \frac{1}{x} = \lim_{x \to 0} \frac{1}{x-\ln(1+x)}(x\ln(1+x))
\]
\[
= \lim_{x \to 0} \frac{1}{1-((1+(1+x))/((1+x))))}
\]
\[
= \lim_{x \to 0} \frac{1}{(x+1)(1+x)}
\]
\[
= \lim_{x \to 0} x/((1+x)(x+x))
\]
\[
= \lim_{x \to 0} (1/(x+2))
\]
\[
= (1/2)
\]
Question 61.

\[\lim_{x \to 0} \left( \frac{1 - \cos 2x}{x^2} \right) = \lim_{x \to 0} \left( \frac{2 \sin 2x}{2x} \right) = 2\]

Question 62.

\[\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e^{\lim_{x \to 0} \left( \frac{\ln(1 + x)}{x} \right)} = e\]

Question 63.

\[\lim_{x \to \frac{\pi}{2}} \left( \frac{\cot x - \cos x}{x^2} \right) = 0\]

Question 64.

\[\lim_{x \to 0^+} (1 + 2x)^{\frac{1}{x}} = e^2\]

Question 65.

\[\lim_{x \to 0^+} \left( \frac{\ln \tan 2x}{\ln \sin 3x} \right) = \lim_{x \to 0^+} \left( \frac{\ln \left( \frac{\tan 2x}{2x} \right)^2}{\ln \left( \frac{\sin 3x}{3x} \right)^3} \right) = 1\]

Question 66.

\[\lim_{x \to 0} \left( e^{x} + x \right)^{\frac{1}{x}} = e^{\lim_{x \to 0} \left( \frac{\ln(e^x + x)}{x} \right)} = e^2\]

Question 67.

\[\lim_{x \to \frac{\pi}{2}^-} \cos 3x \sec 7x = \lim_{x \to \frac{\pi}{2}^-} \left( \frac{\cos 3x}{\cos 7x} \right) = \frac{3}{7}\]

Question 68.

\[\lim_{x \to \infty} \left( 1 + \frac{3}{x} \right)^{2x} = \lim_{x \to \infty} \left( 1 + \frac{3}{x} \right)^{\frac{3}{x}^2} = e^6\]
Question 69.
\[
\lim_{x \to 0} \frac{(e^x - 1)}{(\sin x)} = \lim_{x \to 0} \frac{(e^x)}{(\cos x)} = 1
\]

Question 70.
\[
\lim_{x \to 0^+} \frac{(\ln x)}{(\csc x)} = \lim_{x \to 0^+} \frac{(1/x)}{(-((\cos x)/(\sin^2 x)))} = -\lim_{x \to 0^+} \frac{(\sin^2 x)}{(x \cos x)} = -\lim_{x \to 0^+} \frac{((\sin x)/x)(\sin x)/(\cos x)}{tan x} = 0
\]

Question 71.
\[
\lim_{x \to 0^+} x^{(1/(\ln x))} = e^{\lim_{x \to 0^+} \frac{(\ln x)}{(\ln x)}} = e
\]

Question 72.
\[
\lim_{x \to 0^+} (\cot x - \ln x) = \lim_{x \to 0^+} \frac{(1-\tan x \ln x)}{(\tan x)} = \lim_{x \to 0^+} \frac{(-\sec^2 x \ln x-((\tan x)/x))}{(\sec^2 x)} = \lim_{x \to 0^+} (-\ln x + \cos^2 x) = \infty
\]

Question 73.
\[
\lim_{x \to 0^+} (\sin x)^x = e^{\lim_{x \to 0^+} x \ln (\sin x)} = e^{\lim_{x \to 0^+} \frac{(\ln (\sin x))}{((1/x))}} = e^{\lim_{x \to 0^+} \frac{(((\cos x)/(\sin x)))/(-((1/(x^2))))}{(x \cos x)/1}} = e^{\lim_{x \to 0^+} -((x/(\sin x)))(\cos x)/1}} = e^{\lim_{x \to 0^+} -((x/(\sin x)))(\cos x)}} = 1
\]

Question 74.
\[
\lim_{x \to (\pi/2)^-} (\tan x - \sec x) = \lim_{x \to (\pi/2)^-} \frac{(\sin x - 1)}{(\cos x)} = \lim_{x \to (\pi/2)^-} \frac{(\cos x)}{(-\sin x)} = 0
\]
\[
\lim_{x \to (\pi/2)^-} \frac{(\sin x - 1)}{(\cos x)} = \lim_{x \to (\pi/2)^-} \frac{((-\cos^2 x)/(\cos x)(\sin x + 1))}{(\cos x)/(\sin x + 1))} = 0
\]
\[ \lim_{x \to (\pi/2)^-}(\tan x - \sec x) = \lim_{x \to (\pi/2)^-}((\sin x - 1)/(\cos x))((\sin x + 1)/(\sin x + 1)) = \lim_{x \to (\pi/2)^-}(-\cos^2 x)/(\cos x(\sin x + 1)) = \lim_{x \to (\pi/2)^-}(-\cos x)/(\sin x + 1) = 0 \]

**Question 75.**

\[ \lim_{x \to 0}(x - \sin x)/(x - \tan x) = \lim_{x \to 0}((x - \sin x)/(x - \tan x)) = \lim_{x \to 0}((1 - \cos x)/(1 - \sec^2 x)) = \lim_{x \to 0}((\sin x)/(-2(\sin x)/(\cos x))) = \lim_{x \to 0}((\cos x\sin x)/(-2\sin x)) = \lim_{x \to 0}((\cos x)/(-2)) = -(1/2) \]

**Question 76:** Prove (or disprove) by definition, the following limits:

\[ \lim_{x \to 2}(4x - 5) = 3 \]

**Answer:**

\[ \forall \epsilon > 0, \exists \delta > 0, \quad 0 < |x - 2| < \delta \Rightarrow |(4x - 5) - 3| < \epsilon \]

\[ |(4x - 5) - 3| < \epsilon \Leftrightarrow |4x - 8| < \epsilon \Leftrightarrow 4|x - 2| < \epsilon \Leftrightarrow |x - 2| < \left(\frac{\epsilon}{4}\right) = \delta \]

so if \( \epsilon = 0.001 \) then \( \delta = \left(\frac{0.001}{4}\right) = 0.00025 \)

**Question 77:** Prove (or disprove) by definition, the following limits:

\[ \lim_{x \to -2}(2 + 5x) = -8 \]

**Answer:**

\[ \forall \epsilon > 0, \exists \delta > 0, \quad 0 < |x + 2| < \delta \Rightarrow |(2 + 5x) - (-8)| < \epsilon \]

\[ |(2 + 5x) - (-8)| < \epsilon \Leftrightarrow |5x + 10| < \epsilon \Leftrightarrow 5|x + 2| < \epsilon \Leftrightarrow |x + 2| < \left(\frac{\epsilon}{5}\right) = \delta \]

so if \( \epsilon = 0.002 \) then \( \delta = \left(\frac{0.002}{5}\right) = 0.0005 \)

**Question 78:** Prove (or disprove) by definition, the following limits:

\[ \lim_{x \to 1}(x^2 - 5) = -4 \]
Answer

∀ε > 0, ∃δ > 0.0 < | X 1 | < δ⇐ l ( x² -5 ) - (- 4 ) l < ε
l ( x² -5 ) - (- 4 ) l < ε⇐ l x² -1 l < ε
⇐ l x -1 l | x + 1 | < ε

Seeking " control " over l x + 1 l, majoramos δ < 1 to facilitate accounts (keeping the entire vicinity of the positive side of R )

0 < | X 1 | < δ ⇒ -1 < x , 1 < 1 ≠ 1 ∧ x
⇒ 1 < | x + 1 | < δ ≠ 1 ∧ x
⇒ 1 < | x + 1 | < δ ≠ 3 ∧ x

We use the upper bound ( as we are advancing the mutual implication)

| X - x + 1 || 1 | < ε⇐ | x -1 || 3 < ε
⇐ | x + 1 | < ( ε / 3 ) = δ

choosing δ = min { 1 , ( ε / 3 ) } the implication is rebuilt . Soon ε = 0.01 then δ = min { 1 , (0.01) / 3 ) ) = 0.0033

**Question 79:** Prove (or disprove ) by definition , the following limits : \( \lim_{ x \to -4 } ( 2x + 7) = - 1 \)

Answer:

∀ε > 0, ∃δ > 0.0 < | x - (- 4 ) | < δ⇐ l ( 2x + 7 ) - (- 1 ) l < ε
l ( 2x + 7 ) - (- 1 ) l < ε⇐ l 2x + 8 l < ε
⇐ 2 | x + 4 | < ε
⇐ l x + 4 l < ( ε / 2 ) = δ

if δ = ( ε / 2 ) the proposition is true then -1 has the required property to the limit ( 2x + 7 ) as x tends to -4 .

**80 - Question:** Prove (or disprove ) by definition , the following limits : \( \lim_{ x \to -2 } ( 7-2x ) = 11 \)

Answer:

∀ε> 0, ∃δ> 0.0 < | x - (- 2) | < δ⇐ l ( 7-2x ) -11 l < ε
l ( 7-2x ) -11 l < ε⇐ l -2x-4 l < ε
⇐ l -2 l | x + 2 l < ε
⇐ l x + 2 l < (ε / 2 ) = δ

if δ = (ε / 2 ) the proposition is true just 11 has the property required for the limit of (7-2x) when x tends to -2.
81 - Question: Prove (or disprove) by definition, the following limits: \( \lim_{x \to 3} \frac{(x^2 - 9)}{(x - 3)} = 6 \)

Answer:

\[
\forall \varepsilon > 0, \exists \delta > 0.0 \quad 0 < |x - 3| < \delta \Rightarrow \left| \frac{(x^2 - 9)}{(x - 3)} - 6 \right| < \varepsilon
\]

\[
= \left| x + 3 \right| \cdot 6 < \varepsilon \Rightarrow \left| x - 3 \right| = \delta
\]

if \( \delta = \varepsilon \) the proposition is true just 6 has the required property to the limit \( \frac{(x^2 - 9)}{(x - 3)} \) as \( x \) tends to 3.

Question 82: Prove (or disprove) by definition, the following limits: \( \lim_{x \to -1} (3 + 2x - x^2) = 0 \)

Answer:

\[
\forall \varepsilon > 0, \exists \delta > 0.0 \quad 0 < |x - (-1)| < \delta \Rightarrow \left| (3 + 2x - x^2) - 0 \right| < \varepsilon
\]

\[
= \left| x + 2 \right| < \varepsilon \Rightarrow -1 < x + 2 < 1 \land x \neq -1
\]

\[
\Rightarrow -6 < x - 3 < -4 \land x \neq -1
\]

\[
\Rightarrow |x + 3| < |12| \land x \neq -1
\]

\[
= \left| x + 1 \right| < 12 < \varepsilon
\]

\[
= \left| x + 1 \right| < |\varepsilon / (12)| = \delta
\]

if \( \delta = (\varepsilon / (12)) \) the proposition is true then 0 has the required property to the limit \( (3 + 2x - x^2) \) as \( x \) tends to -1.

Question 83: Calculate the following limits using theorems: \( \lim_{x \to 5} (3x - 7) \)

Answer:

\[
\lim_{x \to 5} (3x - 7) \lim_{x \to 5} (3x) - \lim_{x \to 5} (7)
\]

\[
= 3 \cdot 5 - 7
\]

\[
= 8
\]
**Question 84**: Calculate the following limits using theorems: \( \lim_{x \to 3} (2x^2 - 4x + 5) \)

*Answer:*

\[
\lim_{x \to 3} (2x^2 - 4x + 5) = 2 \lim_{x \to 3} (x^2) - 4 \lim_{x \to 3} (x) + \lim_{x \to 3} (5) = 11
\]

**Question 85**: Calculate the following limits using theorems: \( \lim_{y \to -1} (y^3 - 2y^2 + 3y - 4) \)

*Answer:*

\[
\lim_{y \to -1} (y^3 - 2y^2 + 3y - 4) = -10
\]

**Question 86**: Calculate the following limits using theorems: \( \lim_{x \to -1} \left( \frac{2x + 1}{x^2 - 3x + 4} \right) \)

*Answer:*

\[
\lim_{x \to -1} \left( \frac{2x + 1}{x^2 - 3x + 4} \right) = \frac{-1}{8}
\]

**Question 87**: Calculate the following limits using theorems: \( \lim_{x \to 2} \sqrt{\left( \frac{5 + 2x}{5 - x} \right)} \)

*Answer:*

\[
\lim_{x \to 2} \sqrt{\left( \frac{5 + 2x}{5 - x} \right)} = \sqrt{\left( \frac{14}{9} \right)}
\]

**Question 88**: Calculate the following limits using theorems: \( \lim_{x \to -3} \left( \frac{5 + 2x}{5 - x} \right) \)

*Answer:*

\[
\lim_{x \to -3} \left( \frac{5 + 2x}{5 - x} \right) = \left( \frac{5}{8} \right)
\]
Question 89: Calculate the following limits using theorems: \( \lim_{x \to -3} \left[ 3 \right] \sqrt{\frac{(2x + 5)}{(5-x)}} \)

Answer:
\[
\lim_{x \to -3} \left[ 3 \right] \sqrt{\frac{(2x + 5)}{(5-x)}} = \left[ 3 \right] \sqrt{\lim_{x \to -3} \left( \frac{2x + 5}{5-x} \right)} = \left[ 3 \right] \sqrt{\left( \frac{-1}{8} \right)} = -\frac{1}{2}
\]

Question 90: Calculate the following limits using theorems: \( \lim_{x \to \frac{3}{2}^-} \left( \frac{2x - 3}{4x^2 - 9} \right) \)

Answer:
\[
\lim_{x \to \frac{3}{2}^-} \left( \frac{2x - 3}{4x^2 - 9} \right) = \lim_{x \to \frac{3}{2}^-} \left( \frac{2}{4} \right) \left( \frac{x - \frac{3}{2}}{x^2 - \left( \frac{9}{4} \right)} \right) = \frac{1}{6} \approx 0.16667
\]

Question 91: Calculate the following limits using theorems: \( \lim_{x \to \frac{3}{2}^+} \left( \frac{2x - 3}{4x^2 - 9} \right) \)

Answer:
\[
\lim_{x \to \frac{3}{2}^+} \left( \frac{2x - 3}{4x^2 - 9} \right) = \lim_{x \to \frac{3}{2}^+} \left( \frac{2}{4} \right) \left( \frac{x - \frac{3}{2}}{x^2 - \left( \frac{9}{4} \right)} \right) = \frac{1}{6} \approx 0.16667
\]

Question 92: Calculate the following limits using theorems: \( \lim_{z \to -5} \left( \frac{z^2 - 25}{z + 5} \right) \)

Answer:
\[
\lim_{z \to -5} \left( \frac{z^2 - 25}{z + 5} \right) = z \to \lim_{z \to -5} \left( \frac{(z + 5)(z - 5)}{(z + 5)} \right) = -10
\]

Question 93: Calculate the following limits using theorems: \( \lim_{x \to \frac{1}{3}} \left( \frac{3x - 1}{9x^2 - 1} \right) \)

Answer:
\[
\lim_{x \to \frac{1}{3}} \left( \frac{3x - 1}{9x^2 - 1} \right) = x \to \lim_{x \to \frac{1}{3}} \left( \frac{3}{9x - 1} \right) = \lim_{x \to \frac{1}{3}} \left( \frac{3}{9 (x + \frac{1}{3})} \right) = \frac{1}{2}
\]
**Question 94:** Calculate the following limits using theorems: \( \lim_{x \to 4} \frac{(3x^2 - 17x + 20)}{(4x^2 - 25x + 36)} \)

**Answer:**

\[
\lim_{x \to 4} \frac{(3x^2 - 17x + 20)}{(4x^2 - 25x + 36)} = \lim_{x \to 4} \frac{(3- (4 - x) (x- (5/3 )))}{(4- (4 - x) (x- (9/4 )))}
\]  

\[= 1\]

**Question 95:** Calculate the following limits using theorems: \( \lim_{s \to 1} \frac{(S^3 - 1)}{(s - 1)} \)

**Answer:**

\[
\lim_{s \to 1} \frac{(S^3 - 1)}{(s - 1)} = \lim_{s \to 1} \frac{(s - 1) (s^2 + s + 1)}{(s - 1)}
\]  

\[= 3\]

**Question 96:** Calculate the following limits using theorems: \( \lim_{t \to (3/2)} \sqrt{\frac{(8t^3 - 27)}{(4t^2 - 9)}} \)

**Answer:**

\[
\lim_{t \to (3/2)} \sqrt{\frac{(8t^3 - 27)}{(4t^2 - 9)}} = \sqrt{\lim_{t \to (3/2)} \frac{(8t^3 - 27)}{(9 - 4t^2)}}
\]  

\[= \sqrt{\frac{8}{4} \left( t^3 - \frac{27}{8} \right) \left( t^2 + \frac{9}{4} \right)}
\]  

\[= \sqrt{(9/2)}\]

**Question 97:** Calculate the following limits using theorems: \( \lim_{h \to -1} \frac{\sqrt{h + 5} - 2}{h + 1} \)

**Answer:**

\[
\lim_{h \to -1} \frac{\sqrt{h + 5} - 2}{h + 1} = \lim_{h \to -1} \left( \frac{\sqrt{h + 5} - 2}{h + 1} \right) \left( \frac{\sqrt{h + 5} + 2}{\sqrt{h + 5} + 2} \right)
\]  

\[= (1/4)\]
98 - Question: Calculate the following limits using theorems: \[ \lim_{x \to 1} \left( \frac{\sqrt[3]{x} - 1}{x - 1} \right) \]

Answer:
\[
\lim_{x \to 1} \left( \frac{\sqrt[3]{x} - 1}{x - 1} \right) = \lim_{x \to 1} \left( \frac{\sqrt[3]{x} - 1}{\sqrt[3]{x}^3 - 1} \right) \\
= \lim_{x \to 1} \left( \frac{1}{\sqrt[3]{x}^2 + \sqrt[3]{x} + 1} \right) \\
= \frac{1}{3}
\]

Question 99: Calculate the following limits using theorems: \[ \lim_{x \to 1} \left( \frac{x^3 - x^2 - x + 10}{x^2 + 3x + 2} \right) \]

Answer:
\[
\lim_{x \to 1} \left( \frac{x^3 - x^2 - x + 10}{x^2 + 3x + 2} \right) = \lim_{x \to 1} \left( \frac{(x + 2)(x^2 - 3x + 5)}{(x + 2)(x + 1)} \right) \\
= \lim_{x \to 1} \left( \frac{x^2 - 3x + 5}{x + 1} \right) \\
= \frac{3}{2}
\]

Question 100: Calculate the following limits: \[ \lim_{t \to 2^-} \left( \frac{-t + 2}{(t-2)^2} \right) \]

Answer:
\[
\lim_{t \to 2^-} \left( \frac{-t + 2}{(t-2)^2} \right) = \lim_{t \to 2^-} \left( \frac{-1}{(t-2)} \right) \\
= +\infty
\]

Question 101: Calculate the following limits: \[ \lim_{x \to 0^+} \left( \sqrt{3 + x^2} \right) \]

Answer:
\[
\lim_{x \to 0^+} \left( \sqrt{3 + x^2} \right) = +\infty
\]

Question 102: Calculate the following limits: \[ \lim_{x \to 0} \left( \sqrt{3 + x^2} \right) \]

Answer:
\[
\lim_{x \to 0} \left( \sqrt{3 + x^2} \right) = +\infty
\]

Question 103: Calculate the following limits: \[ \lim_{x \to 4^-} \left( \sqrt{16 - x^2} \right) \]

Answer:
\[
\lim_{x \to 4^-} \left( \sqrt{16 - x^2} \right) = -\infty
\]
Question 104: Calculate the following limits: \( \lim_{x \to 0^+} \left( \frac{x^2 - 3}{x^3 + x^2} \right) \)

Answer:
\[
\lim_{x \to 0^+} \left( \frac{x^2 - 3}{x^3 + x^2} \right) = \lim_{x \to 0^+} \left( \frac{x^2(x + 1)}{x^2(x + 1)} \right) = -\infty
\]

Question 105: Calculate the following limits: \( \lim_{s \to 2^-} \left( \frac{1}{s-2} - \frac{3}{s^2-4} \right) \)

Answer:
\[
\lim_{s \to 2^-} \left( \frac{1}{s-2} - \frac{3}{s^2-4} \right) = \lim_{s \to 2^-} \left( \frac{s+2}{2} \right) = -\infty
\]

Question 106: Calculate the following limits: \( \lim_{x \to 1^-} \left( \frac{2x^3 - 5x^2}{x^2 - 1} \right) \)

Answer:
\[
\lim_{x \to 1^-} \left( \frac{2x^3 - 5x^2}{x^2 - 1} \right) = +\infty
\]

Question 107: Calculate the following limits: \( \lim_{x \to 1^-} \left( \frac{[x^2] - 1}{x^2 - 1} \right) \)

Answer:
\[
\lim_{x \to 1^-} \left( \frac{[x^2] - 1}{x^2 - 1} \right) = +\infty
\]

Question 108: Calculate the following limits: \( \lim_{x \to -2^-} \left( \frac{6x^2 + x - 2}{2x^2 + 3x - 2} \right) \)

Answer:
\[
\lim_{x \to -2^-} \left( \frac{6x^2 + x - 2}{2x^2 + 3x - 2} \right) = \lim_{x \to -2^-} \left( \frac{(x + (2/3))(x - (1/2))}{(2 + x)(x - (1/2))} \right) = +\infty
\]

Question 109: Calculate the following limits: \( \lim_{x \to 2^-} \left( \frac{x - 2}{2 \sqrt{4x - x^2}} \right) \)

Answer:
\[
\lim_{x \to 2^-} \left( \frac{x - 2}{2 \sqrt{4x - x^2}} \right) = -\infty
\]
Unit 2. Differential calculus

Introduction to the Unit

This unit introduces the concept of continuity, the derivative functions in the geometric interpretation derived, the exterior derivative and differential forms of functions. These are fundamental concepts of differential calculus and enable register information about functional relationships in R, in the form of differential equations.

Unit Objectives

By the end of this unit, you should be able to:

determine whether a real function is continuous or not a point or in a range, using the theorem of Bolzano or intermediate value as to whether or not there is zero a function within a certain range;

calculate derived elementary functions using the theorems of derived functions composed, implicit function and inverse function.

determine local and global extremes determine the monotony and the curvature of real functions graphics, with the help of the theorems of Fermat, Cauchy, Rolle, Lagrange and l’Hospital.

Key Terms

- Fun. continuous: If \( a \in \text{dom} f \) limit exists and is equal to \( f(a) \)
- Cont. right: If limit right to exist and is equal to \( f(a) \)
- Cont. Left: If limit left in the exists and is equal to \( f(a) \)
- Cont. Range: Continuous anywhere in the range
- \( C^0 (e) \): Set all functions. continuous in \( E \subset \mathbb{R} \)
- Fun. Dirichlet: 1 if \( x \in \mathbb{Q} \) and 0 if \( x \in \mathbb{R} \setminus \mathbb{Q} \) (descont todolugar.)
- Descont. simple: If any side limit limit exists in the
- Descont. endless: If any lateral boundary limit does not exist in the
- Oscillation \( f \) at \( e \): \( \omega (f, E) = \sup \{ f(E) \} - \inf \{ f(E) \} \)
**Oscillation** \( f(x) \): \( \omega(f, x) = \inf \{ \text{diam} f(V_\varepsilon(x) \cap \text{dom} f), \varepsilon > 0 \} \)

**Unif. Cont. At** \( \forall \varepsilon > 0, \exists \delta > 0: (\forall y \in A \land |xy| < \delta) \Rightarrow |f(x) - f(y)| < \varepsilon \)

**Compact subc**: closed and delimited \( \mathbb{R} \)

**Fun. Increasing**: \( x \leq y \Rightarrow f(x) \leq f(y) \)

**Fun. Decreasing**: \( x \leq y \Rightarrow f(x) \geq f(y) \)

**Fn. Lipschitz cont**: \( |f(x) - f(y)| \leq k |x - y|, \forall x, y \in \text{dom} f, \) for some \( k > 0 \)

**Derivative of** \( f \): \( f'(x_0) \lim_{h \to 0} \frac{(f(x_0 + h) - f(x_0))}{h} \)

**Fn. Differentiable**: In \( x \in \) the derivative exists

**Chain rule**: \( (g \circ f)'(x) = g'(f(x)) \cdot f'(x) \)

**Fees related**: If \( E(x(t), y(t)) \) then \( E = c'(x(t), y(t)) = 0 \)

**Polynomial**. Taylor \( f(x_0) + (f'(x_0) / (1!))(x-x_0) + (f''(x_0)) / (2!)(x-x_0)^2 + \cdots \)

**Local maxima** If there \( V_\varepsilon(c) \) where \( f(c) \geq f(x) \)

**Local minima** If there \( V_\varepsilon(c) \) where \( f(c) \leq f(x) \)

**Global minimum** If there \( c \in I \) where \( f(c) \leq f(x) \) \( \forall x \in I \)

**Critical point** \( f \): \( f'(c) = 0 \) and \( f''(c) \) there is no

**Inflection point**: derivative changes sign

**Increment** \( y \) : \( \Delta y \)

**Differentially** If \( y = f(x) \) then \( dy = f'(x) \Delta x \)

**Notation of leibniz** \( F'(x) = dy / dx \)

**Exterior derivative**: \( Df = f'(x) dx \)

**Differential equation**: relationship between variables of differential
Activity 1 - Continuity functions

Introduction

Continuity of real functions of real variable. Some local properties of continuous functions. Alternatively continuity characterization. Notable examples of continuous functions. Discontinuity functions. Global continuous functions properties (Theorem of Bolzano or intermediate value)

Activity Details

A function \( f \colon \mathbb{R} \to \mathbb{R} \) is continuous at the point \( a \in \mathbb{R} \) iff

(i) \( f(a) \in \mathbb{R} \), the function is defined in point;

(ii) \( \lim_{x \to a} f(x) \in \mathbb{R} \), the function limit at point \( a \) exists;

(iii) \( f(a) = \{x \to a \lim\} \ f(a) \)

the three conditions together are equivalent to the following statement:

\[
\forall \varepsilon > 0, \exists \delta > 0: |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon
\]

While the limit definition in point is only necessary that is an accumulation point \( \text{dom} f \), the definition of continuity at a point needs to \( a \in \text{dom} f \).

If \( f(a) \) exists, \( \lim_{x \to a^-} f(x) \) exists \( f(a) = \{x \to a^- \lim\} \ f(a) \) said that the function is continuous on the right of \( a \).

If \( f(a) \) exists, \( \lim_{x \to a^+} f(x) \) exists \( f(a) = \{x \to a^+ \lim\} \ f(a) \) it is said that the function is continuous to the left of it.

A continuous function at a point has to be continuous to the right \( (a^+) \) and left \( (a^-) \).

A function is continuous in an interval \([\text{open } a, b] \subset \mathbb{R}\) iff is continuous at every point of that range.

A function is continuous in a closed interval \([a, b] \subset \mathbb{R}\) if is continuous at all points of the interval \( a, b \)[, continued to continuous and from right to left \( b \).

Corresponding definitions are given for continuity functions in mixed ranges.

The set of all continuous functions in \( E \subset \mathbb{R} \) subset is represented by \( C(E) \) or \( C^0(E) \).

Sometimes it is taught that a function is continuous if its graph can be drawn without lifting the pencil from the paper. Despite being an “illustrative” of much of continuous functions, this definition “technically” is not valid. For example, the function \( f \colon \mathbb{N} \to \mathbb{R} \) given by the general term \( f(x) = x \) is a continuous function at any point \( n \) of your domain and your chart can be drawn up pencil paper points!
Why is this continuous function? Applying the definition for any $\varepsilon > 0$ the value $\delta = (1/2) > 0$ satisfies the condition of continuity at any point $n \in \mathbb{N}$

$$|x_n| < (1/2) \Rightarrow (x = n) \Rightarrow |f(x) - f(n)| = 0 < \varepsilon$$

Moreover, as the statement was not necessary to specify the function $f$, any function $f: \mathbb{N} \to \mathbb{R}$ is continuous.

The continuity of a function at a point preserves the limit:

If $\lim_\rightarrow a \{ h(x) \} = b$ is a continuous function in $b$ then

$$f(\lim_\rightarrow \{ x \to a \} h(x)) = \lim_\rightarrow \{ x \to a \} f(h(x))$$

the proof follows these lines:

Since $f$ is continuous then $b >$

$$\forall \varepsilon_1 > 0, \exists \delta_1 > 0 : |y_b| < \delta_1 \Rightarrow |f(y) - f(b)| < \varepsilon_1$$

On the other hand, as the $\lim_\rightarrow \{ x \to a \} h(x) = b$ is that in particular for

$$\delta_1 > 0, \exists \delta_2 > 0 : |x| < \delta_2 \Rightarrow |h(x) - b| < \delta_1$$

the composition in the neighborhood it follows that $y = h(x)$ for some $x$ apparent that

$$\forall \varepsilon_1 > 0, \exists \delta_1 > 0 : |h(x) - b| < \delta_1 \Rightarrow |f(h(x)) - f(b)| < \varepsilon_1$$

of the two preceding conditions obtain was found

$$\forall \varepsilon_1 > 0, \exists \delta_2 > 0 : |x| < \delta_2 \Rightarrow |f(h(x)) - f(b)| < \varepsilon_1$$

Hence it follows that

$$\lim_\rightarrow \{ x \to a \} f(h(x)) = f(b)$$

$$\lim_\rightarrow \{ x \to a \} f(h(x)) = f(\lim_\rightarrow \{ x \to a \} h(x))$$

Continuity Characterization:

Topological: a real function $f$ is continuous iff pre-image an open is an open

Metric: the real function $f$ is continuous at the point iff $f$ is defined on some open interval containing and for any $\varepsilon$ neighborhood of $f(a)$ there is a $\delta$ neighborhood of $a$ such that $f(V_\{} \{ \varepsilon \} (a)) \subset V_\{} \{ \varepsilon \} (f(a))$
Sequential: a real function $f$ is continuous at the point $a \in \mathbb{R}$ iff for any sequence $(x_n)$ in dom $f$, converging to the sequence $(f(x_n))$ in img $f$ converges to $f(a)$:

$$\lim_{n \to \infty} x_n = x \iff \lim_{n \to \infty} f(x_n) = f(x)$$

Ex: The Dirichlet function is defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

It is discontinuous at any point because it $a \in \mathbb{R}$ ($x_n$) is a sequence in $\mathbb{Q}$ and $(y_n)$ is a sequence in $\mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to a$ and $y_n \to a$ to have that $f(x_n) \to 1$ and $f(y_n) \to 0$.

As a corollary of the conservation limits theorem it follows that if there is a continuous function in $a \in \mathbb{R}$ a continuous function $h(a)$ then the composite $f \circ h$ function is continuous in the domain is made as follows:

If there is a continuous function in the then $\lim_{x \to a} h(x) = h(a)$, as $f$ is continuous at $h(a)$ applying the above proposition is obtained:

$$\lim_{x \to a} (f \circ h)(x) = x \to \lim_{a} f(h(a)) = f(h(a)) = (f \circ h)(a)$$

immediately $f \circ h$ is continuous in the notable examples of continuous functions:

Any polynomial $p(x)$ is continuous around $\mathbb{R}$;  
Any rational function $f(x) = (p(x)) / (q(x))$ is continuous on its domain (note that the excepted area where $q(x) = 0$),  
the function $f(x) = [n] \sqrt{x}$ is continuous in its domain ($\mathbb{R}$ if $n$ is odd and $\mathbb{R} \setminus \{> 0\}$ if $n$ is even);  
The function $f(x) = \sin(x)$, $f(x) = \cos(x)$ are continuous throughout $\mathbb{R}$;  
The functions $f(x) = \tan x$ is continuous in its domain,  
the function exponential $f(x) = e^x$ is continuous throughout $\mathbb{R}$;  
The logarithmic function $f(x) = \ln x$ is continuous at $\mathbb{R} \setminus \{> 0\}$ once $\text{img}(e^x) = \mathbb{R} \setminus \{> 0\}$ $x$ is continuous $\mathbb{R}$

**Discontinuity functions**

If a function is not continuous in point of its dom$\subset \mathbb{R}$ domain, it is said that it is discontinuous at this point, i.e. there is an impenetrable neighborhood centered at $f(a)$, whatever approach you to do $a \in$ in the field:

$$\exists \varepsilon > 0, \forall \delta > 0: \forall x \in \text{dom} f \setminus | x | < \delta \Rightarrow | f(x) - f(a) | \geq \varepsilon$$

the discontinuity points of a function classified according to the presence or absence of lateral limits $f$. 
If $R \rightarrow R$ is a discontinuous function in $a \in \text{dom} f$ then it is said that is a discontinuity point:

The first kind or simple when the lateral limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist;

Second species or infinite when at least one of the lateral boundaries $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ does not exist;

A discontinuity of first kind is removable when the $\lim_{x \to a} f(x)$ and there is a jump when the lateral boundaries exist but are different.

The oscillation of a real function in $E \subseteq R$ subset is defined as the difference between the highest and the smallest of $f$ in $E$

$$\omega(f; E) = \sup f(E) - \inf f(E)$$

the oscillation $\omega(f; x)$ a real function function $f$ at a point $x$ is defined as

$$\omega(f; x) = \inf \{\text{diam } f(V_\varepsilon(x) \cap \text{dom} f), \varepsilon > 0\}$$

where $\text{diam } f(V_\varepsilon(x) \cap \text{dom} f) = \sup \{|ab|: a, b \in V_\varepsilon(x) \cap \text{dom} f\}$

Alternatively, a real function $f$ is continuous at $x_0$ iff your swing at this point is zero, $\omega(f; x_0) = 0$

Global Continuous Functions Properties

A real function $f$ is uniformly continuous on a subset $A \subseteq \text{dom} f$ if the following condition is met:

$$\forall \varepsilon > 0, \exists \delta > 0: (\forall y \in A \land |xy| < \delta) \Rightarrow |f(x) - f(y)| < \varepsilon$$

The uniform continuity depends on $f$ $\subseteq \text{dom} f$ subset $\forall \varepsilon$.

NOTE: a continuous function at all points of $A \subseteq \text{dom} f$ satisfies the condition:

$$\forall y \in A, \delta>0, \exists \delta > 0: |xy| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

The difference between the two definitions lies “only” in exchange for the position of quantifiers ... $\exists \delta, \forall y \ldots$ (there is a $\delta$ for any $y$) to ... $\forall y, \exists \delta \ldots$ (for any $y$ there is some $\delta$)

Ex: a function may be continuous at all points of $A \subseteq \text{dom} f$ without being uniformly continuous in $A$. the function $f(x) = x^2$ is continuous at any point $R$ but not uniformly continuous across the $R$: Taking $\varepsilon = n x = (\delta / 2)$ then $|xy| < \delta$ but can always find a $\delta$ such that $|f(x) - f(y)| = n\delta + ((\delta / 2))^2 > \varepsilon$, for any given $\varepsilon$, simply by choosing a sufficiently large $n$, which is secured by property . Archimedean of $R$

This is because $R$ is not compact

A subset of $R$ is compact if and only if it is closed and bounded $f$...

(Continuity preserves compactness) If $K \subseteq R$ is a compact set $K \to R$ is a continuous function then $f(K)$ is a compact $f$ is uniformly continuous in $K$.

The monotony of a function defined similarly to the sequences set to (are also functions):

$x \leq y \Rightarrow f(x) \leq f(y)$ increasing

$x \leq y \Rightarrow f(x) \geq f(y)$ decreasing
Be \( I \subset \mathbb{R} \) a range nondegenerate \( f: I \to \mathbb{R} \). If \( f \) is a continuous and injective function then \( f \) is monotonic, the inverse function exists and also \( f^{-1}: \text{im} f \to \mathbb{R} \) is continuous and injective functions:

Some notable theorems on continuous

(Bolzano or intermediate value) if the function \( f \) is continuous on \([a, b]\) interval, then \( f(x) \) takes all values between \( f(a) \) and \( f(b) \) \( \forall k: (f(a) \leq k \leq f(b)) \Rightarrow a \leq c \leq b \land f(c) = k \)

that is, for any \( k \) number between \( f(a) \) and \( f(b) \) there will be a number between \( c \) and \( b \) such that \( f(c) = k \).

(Weierstrass) If \( [a, b] \to \mathbb{R} \) is a continuous function, then \( f \) has maximum and minimum points in \([a, b] \).

(Brower) If \( [a, b] \to [a, b] \) then \( f \) has a fixed point (a point \([a, b]\) where \( f(x) = x \))

(Preserves connectivity) If \( I \subset \mathbb{R} \) is a connected set \( f: I \to \mathbb{R} \) is a continuous function, then \( f(I) \) is a connection.

(Preserves algebraic operations) If \( fg \) are continuous functions on a point, then

\( f + g \) is continuous at \( a \);

\( fg \) is continuous at \( a \);

\( fg \) is continuous in the

\( f/g \) is continuous in \( a \), since \( g(a) \neq 0 \)

The set \( C(\mathbb{R}) \) of all continuous functions in \( \mathbb{R} \) is a vector space over \( \mathbb{R} \) and is also an algebra (with the multiplication operation functions).

A real function is Lipschitz continuous iff there is a number \( k > 0 \) such that

\[ |f(x) - f(y)| \leq k |xy|, \forall x, y \in \text{dom} f \]

If \( f \) is continuous in the Lipschitz then \( f \) is uniformly continuous in \( A \).

A is a real function contraction iff there are a number \( \alpha \in (0, 1) \) [such that

\[ |f(x) - f(y)| \leq \alpha |xy|, \forall x, y \in \text{dom} f \]

(Banach) If a real function \( f \) is a contraction such that \( f([a, b]) \subset [a, b] \), then there is a single number \( \alpha \in [a, b] \) which is the fixed point of \( f \). More precisely, given \( x_0 \), the sequence \( (x_n) \) recursively defined by \( x_{n+1} = f(x_n) \), \( \forall n \in \mathbb{N} \) converges to.

Conclusion

This unit analyzed the important concept of continuity of functions actual real variable. Some local properties of continuous functions. Alternatively continuity characterization. Notable examples of continuous functions. Discontinuity functions. Global continuous functions properties (Theorem of Bolzano or intermediate value)
Formative Assessment

Activity Assessment 1 (with answers)

Group a minimum of 5 and a maximum of 10 of the following questions in every 1 hour assessment test without consultation.

Select questions - type different in each test preparation, and changing the parameter database to avoid gluing

Each question is worth between 10% and 20%, depending on the number of selected questions Q:

-3−H(x) = -∞ and lim_{x→-3} H(x) = +∞)

3 - Question: Determine function of discontinuities: h(x) = ((x + 1) / (2x + 5))

Answer: is discontinuous x = -(5/2) because it is not defined and is continuous in all other real numbers.

4 - Determine the discontinuities function

f(x) = ((x + 2)^2 is x≤0 and x^2 + 2 if x>0

Answer: Discontinuous.

1 - Determine the discontinuities function:

G(x) = {((x^2-3x-4) / (x-4)) if x ≠ 4, and 2 x = 4

Answer: Discontinuous at x = 4 because lim_{x→4} ((x^2-3x-4) / (x-4)) = lim_{x→4} (x + 1) = 5 ≠ L(4) = 2

2 - Question: Determine function of discontinuities H(x) = ((x^2-5x + 4) / ((x-1)(x²-x-12)))

A. discontinuity at x = 1 because the function is not defined at this point (1∈domf) and at x = -3 since lim_{x→-3} H(x) does not exist. (lim_{x→0} x + 2) = 4 and lim_{x→0} x^2 + 2 = 2. Is continuous at R \ {0}

5 - Question: Determine the function of discontinuities:

f(x) = 1 if x≥1

f(x) = -1 if x<1

A. is discontinuous at x = 1

6 - Question: Prove that if f is continuous at x = a then lim_{x→a} f(x) = f(a)

Answer: the possibilities for g is discontinuous at x = a is:

because g(x) is not set then x = a, then (f + g)(x) = f(x) + g(x) is also not defined in a.

because lim_{x→a} f(x) does not exist at x = a, then the lim_{x→a} (f + g)(x) = lim_{x→a} f(x) + lim_{x→a} g(x) does not exist in the.

because the gas threshold (a) It exists but is not equal to lim_{x→a} f(x), then (f + g)(a) is defined in a, but will not equal the lim_{x→a} f(x) + lim_{x→a} g(x)
7 - Q: Determine the continuity of the composite function $f \circ g$

Answer:

$$f (x) = \sqrt{g (x)} = 16 + x^2$$

$$(f \circ g) (x) = \sqrt{(16 + x^2)}$$

$\text{gift } (f \circ g) = \{x \in \mathbb{R}: 16 + x^2 \geq 0\} = \mathbb{R}$$

The function is continuous in its domain.

8 - Question: Determine the continuity of the composite function $f \circ g$

Response:

$$f (x) = \frac{1}{(x^2)} g (x) = x + 3$$

$$(f \circ g) (x) = \frac{1}{(x + 3)^2}$$

$\text{gift } (f \circ g) = \{x \in \mathbb{R}: x + 3 \neq 0\} = \mathbb{R} \setminus \{-3\}$$

The function is continuous in its domain.

9 - Question: Determine continuity of the function $f (r) = \frac{(r + 3)}{(r^2-4)}$

Answer:

$$f (r) = \frac{(r + 3)}{(r^2-4)}$$

$$= \{\text{dom } f \in \mathbb{R}: r^2 - 4 \neq 0\} = \mathbb{R} \setminus \{-2, 2\}$$

in $]0, 4]$ the function is not continuous at $2$

in $]-2.2[$ the function is continuous

in $]-\infty, -2]$ the function is not continuous at $-2$

in $]2, + \infty[$ the function is continuous

in $[-4, 4]$ the function is not continuous in $\{-2, 2\}$

10 - Question: Determine continuity of the composite function $f \circ g$

Answer:

$$f (x) = \sqrt{x} \quad g (x) = 16 + x^2$$

$$(f \circ g) (x) = \sqrt{(x^2-16)}$$

$$\text{gift } (f \circ g) = \{x \in \mathbb{R}: 16 - x^2 \geq 0\} = [-4, 4]$$

The function is continuous in $[-4, 4]$

11 - Question: Determine continuity of the composite function $f \circ g$
Resposta:

\[ f(x) = \sqrt[3]{x}; g(x) = \sqrt{x+1} \]

\[ (f \circ g)(x) = \sqrt[3]{\sqrt{x+1}} \]

\[ \text{dom}(f \circ g) = \{ x \in \mathbb{R} : x + 1 \geq 0 \} = [-1, +\infty[ \]

A function is continuous in \([-1, +\infty[ \]

12. Question: Be \( 0 \leq f(x) \leq 1 \) for \( 0 \leq x \leq 1 \). Prove that if \( f \) is continuous at \([0.1]\), there are a number \( c \in [0,1] \) such that \( f(c) = c \)

A.

If \( f(0) = 0 \) is proved

If \( f(1) = 1 \) it is also proved

Suppose then that \( f(0) > 0 \) and \( f(1) < 1 \)

consider the function \( g(x) = f(x) - x \)

Since \( f(0) < 0 \) then \( g(0) = f(x) - x < 0 \)

As \( F(0) > 0 \) then \( g(0) = f(x) - x > 0 \)

The function \( g(x) \) is continuous in \([0,1]\)

Therefore there is a \( c \in [0,1] \) where \( g(c) = 0 \)

other words \( f(c) = c \)

13 - Question:

\( f(x) = x + x^3 + 3 \) is continuous in the range \([-2, -1]\)

A.

The values of \( f(-2) = -7 \quad f(-1) = 1 \)

Therefore the theorem Bolzano there is a \( c \in \) value \([-2, -1]\) where \( f(c) = 0 \)

### Activity 2 - Derivative functions in R

**Introduction**

Activity Details

The approach of a continuous function of problem is often found in scientific research in various fields.

If there are no more information about a function other than the function value $f(x_0)$, a point $x_0 \in [a, b]$, that information can be extrapolated on the basis of value in $x_0$ neighborhood?

The method of linear approximation, uses the line $g(x) = \alpha \cdot x + \beta$, to address this issue.

Changing to a local variable $h = x - x_0$ in $x_0$ neighborhood, you write the line that will be used to approximate $f(x)$ in $x_0$ neighborhood as $g(h) = + \alpha \cdot h$ be one imposes two conditions:

1. $g(0) = f(x_0)$

2. $\lim_{h \rightarrow 0} (f(x_0 + t) - g(h)) = 0$

The first condition is equivalent $ab = f(x_0)$

The second condition is equivalent the

$\lim_{h \rightarrow 0} [(f(x_0 + h) - f(x_0)) - \alpha \cdot h] = 0$

it follows that the slope of the line that approximates $f(x)$ the conditions imposed by (1) and (2) is given by

$\alpha = \lim_{h \rightarrow 0} \frac{(f(x_0 + h) - f(x_0))}{h}$

If this limit exists is said that $f(x)$ is differentiable (or derivable) point $x_0$, and this number is called the derivative of the function $f(x)$ at the point $x_0$:

$f'(x_0) = \lim_{h \rightarrow 0} \frac{(f(x_0 + h) - f(x_0))}{h}$

or alternative notations $D_x f(x_0)$ or by $(df / dx)(x_0)$.

If the function is differentiable at all points inside the domain, then there is a derived function

$f'(x) = \lim_{h \rightarrow 0} \frac{(f(x + h) - f(x))}{h}$

whose value at each point is exactly the derivative of $f(x)$ at that point.

Note: the general term of the function $f(x)$ x shows the symbol used for the independent variable function. Thus, $f'(x)$ is calculated with respect to this variable $x$. Ifare given $f(g)$ the derivative $f'(g)$ is calculated on the variable $g$.

Straight line tangent and derivative

The graph of the function $y = f(x)$ is approximated in $x_0$ point, the line with slope $\alpha$ through the point $(x_0, y_0)$

$\lim_{\Delta x \rightarrow 0} [\Delta y - \alpha \cdot \Delta x] = 0$ where $x = \Delta h = x-x_0$, $\Delta y = y-y_0$ and $y_0 = f(x_0)$.

figure down draw up the drying when $x$ approaches $x_0$ on both sides:
The slope of the secant is given by the incremental ratio \( \frac{\Delta y}{\Delta x} \). Successively, the tangent can be approximated by drying on both sides of \( x_0 \) and the slope of the tangent line is given by the threshold:

\[
x \rightarrow \Delta \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \alpha = \Delta \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}
\]

Therefore, if \( f \) is derivable at \( x_0 \), then the linear approximation of \( f \) in \( x_0 \) neighborhood is given by the line tangent to the graph at the point \( (x_0, f(x_0)) \):

\[
y = f(x_0) + f'(x_0)(x - x_0)
\]

The line tangent to the curve at \( x_0 \) point has a slope equal to the derivative of the function \( y = f(x) \) in \( x_0 \) point.

\[
\text{side derivative is defined on the edge of closed intervals [a, b], and the respective side edge of the quotient}
\]

\[
f'(a) = h \rightarrow 0^+ \lim_{h \rightarrow 0^+} \frac{f(a + h) - f(a)}{h} \quad \text{and}
\]

\[
f'(b) = h \rightarrow 0^- \lim_{h \rightarrow 0^-} \frac{f(b + h) - f(b)}{h}
\]

Differentiability and continuity.

Let \( f \) be a real function defined on \([a, b]\). If \( f \) is differentiable at \( x \in [a, b] \) then \( f \) is continuous at that point.

The statement is made as follows: as \( h \rightarrow 0 \) has the algebra limits

\[
\lim_{h \rightarrow 0} f(x + h) - f(x) = h \rightarrow \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}
\]

\[
= \{H \rightarrow 0 \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \cdot h
\]

\[
= f'(x) \cdot h
\]

\[
= 0
\]

Therefore \( f \) is continuous.
The reciprocal is not true.

Ex: The function \( f(x) = |x| \). It is continuous at \( x = 0 \) but is not differentiable at that point.

Geometrically, when the curve is not “smooth” and makes an angle at a point, have different lateral limits at this point and will not be differentiable.

Derivation of algebraic rules

Let \( f \) and \( g \) are real functions defined in \([a, b]\). If \( f \) and \( g \) are differentiable in \( x \in \text{point } [a, b] \) então:

1. \((f+g)'(x)=f'(x)+g'(x)\)
2. \((fg)'(x)=f'(x)g(x)+f(x)g'(x)\)
3. \(((f/g))'(x)=((f'(x)g(x)-g'(x)f(x))/(g^2(x))) \text{ to } g(x) \neq 0\)

The proof is given as follows:

(1) \((f + g)'(x) = \lim_{h \to 0} \frac{((f + g)(x + h) - (f + g)(x))}{h}\)

\lim_{h \to 0} \frac{((f(x + h) - f(x))/h) + \lim_{h \to 0} \frac{((g(x + h) - g(x))/h)}{h}}{h}\)

\(= f'(x) + g'(x)\)

(2) Using the property limits:

\((fg)'(x) = \lim_{h \to 0} \frac{((fg)(x + h) - (fg)(x))/h}{h}\)

\lim_{h \to 0} \frac{((f(x + h) - (f(x))/h)g(x + h) + f(x) [g(h + x) - g(x)]/h)}{h}\)

\(= f'(x)g(x) + f(x)g'(x)\)

(3) Using the property limits:

\(((f/g))'(x) = \lim_{h \to 0} \frac{(((f(x + h) - f(x))/h)g(x + h) - f(x)[g(h + x) - g(x)]/h)}{h}\)

\(= ((f'(x)g(x) - g'(x)f(x))/g^2(x))\)

Derivation of Composite functions.

Let \( f \) be a continuous function in the interval \([a, b]\), differentiable at some point \( x \in [a, b] \) and \( g \) be a continuous function in \( \text{Im} f \) interval and differentiable at \( f(x) \). If the composition \( g \circ f \) is possible in a neighborhood of \( x \) then \( g \circ f \) is differentiable at \( x \) and derivative is given by expression (chain rule)

\((g \circ f)'(x) = g'(f(x)) \cdot f'(x)\)
The proof is given as follows: Since \( f \) is differentiable in \( x \), the linear approximation of \( f \) in the neighborhood of \( x \), where the composition is possible, is given by

\[
f(x + h) - f(x) = [f'(x)] + h \cdot (H)
\]

Note: \((h)\) represents a function class such that for any function \( g \in o(h) \) has \( \lim_{h \to 0} \frac{g(h)}{h} = 0 \).

Using the variable \( y = f(x) \) and \( s \) symbol to the local variable in the vicinity of \( f(x) \), has

\[
g(s + y) - g(y) = [g'(y) + o(s)] \cdot s
\]

is obtained Hence

\[
(g \circ f)(x + h) - (g \circ f)(x) = g(f(x + h)) - g(f(x))
\]

\[
= [g'(y) + o(s)] \cdot f(x + h) - f(x)
\]

\[
= [g'(y) + o(s)] \cdot [f'(x) + o(h)] \cdot h
\]

So

\[
\frac{((g \circ f)(x + h) - (g \circ f)(x))/h}{h} = [g'(y) + o(s)] \cdot [f'(x) + o(h)]
\]

E the limit when \( h \to 0 \) is obtained

\[
(g \circ f)'(x) = g'(y) \cdot f'(x)
\]

which is the desired result as \( y = f(x) \).

Derived from the inverse function

If \( f \) is a differentiable function injective and an open interval \( a, b \), then

\[
(f^{-1})'(f(x)) = \frac{1}{f'(x)} \text{ at } x \in [a, b]
\]

the statement is made as follows: if \( f \) is injective \( f'(x) \neq 0 \) for any point of an open interval.

Furthermore, admits reverse \( f^{-1} : \text{Im} f \to \mathbb{R} \).

So for any \( x \in [a, b] \) that \( x = f^{-1}(f(x)) \).

Applying the chain rule obtain

\[
1 \cdot (f^{-1})'(f(x)) \cdot f'(x)
\]

and the result follows

Example:

Calculate the derivative of the function \( f(x) = \ln(1 + x^2) \). Let \( u = (x^2 + 1) \)

\[
f'(u(x)) = f'(u) \cdot u'(x)
\]

\[
= (1/u)(x^2)
\]

\[
= ((2)/(x^2 + 1))
\]
From the algebraic rules and the chain rule can be calculated derived from simple or compound functions:

c) \( = 0 \) where \( c \) is a constant

\( (x) = 1 \)

\( (x^n) = nx^{n-1} \) to \( n \in \mathbb{Z} \)

\( (u^n) = nu^{n-1}u' \) where \( u(x) \) is a function

\( (\sqrt{u}) = \frac{(u')}{(2\sqrt{u})} \) for \( u(x) > 0 \)

\( (u^r) = ru^{r-1}u' \) to \( r \in \mathbb{Q} \) \( (x) \neq 0 \)

\( \sin u) = u'\cos u \)

\( \cos u)' = -u'\sin u \)

\( (\tan u)' = u'\sec^2u \) in its domain

\( (\sec u)' = u'\tan u\sec u \)

\( (\arcsin u)' = \frac{(u')}{(\sqrt{1-u^2})} \)

\( (\arccos u)' = -\frac{(u')}{(\sqrt{1-u^2})} \)

\( (\arctan u)' = \frac{(u')}{(1 + u^2)} \)

\( (e^x)' = e^x \) (remarkable property)

\( (e^u)' = u'e^u \)

\( (a^u)' = u'a^u \ln a \) for \( a > 0 \)

\( (u^v)' = v'u + vu^{v-1}u' \) to \( v(x) > 0 \) \( v(x) \) function of \( x \)

\( (\ln x)' = \frac{1}{x} \)

\( (\ln u)' = \frac{(u')}{u} \)

**Higher Order Derivatives.**

The derivation may be applied to its own derived function to obtain the second derivative

\[ f'(x) = (f'(x))' \]

the process may be repeated \( n \) times for \( n \)-th derived

\[ f^{(n)}(x) = (f^{(n-1)}(x))' \]

agreed to designate the derivative of order zero to own function

\[ f^{(0)}(x) = f(x) \]
Example

Calculate the second derivative of the function \( f(x) = x^{2x} \)

\[
\begin{align*}
f'(x) &= e^{2x} + 2xe^{2x} \\
f''(x) &= 3e^{2x} + 4xe^{2x}
\end{align*}
\]

**Derivative of Implicit Functions**

The equation \( E(x, y) = 0 \) can explicitly define the function \( y = f(x) \) is variable \( y \) can be solved in terms of \( x \), or implicitly one or several functions that satisfy the equation

\[
is(x f(x)) = 0
\]

the differential operator \( D_{x} \) can be applied to both sides of the equation for \( f'(t) \)

\[
E'(x, f(x)) = 0
\]

Example: Given the equation \( (1/x) + (1/y) = 1 \), where \( x \neq 0, y \neq 0 \), consider \( y = f(x) \) and find \( y' \).

\[
\begin{align*}
(1/x) + (1/y) &= x + 1 \\
\Rightarrow y - xy &= 0 \\
(1-x) y + 1 &= 0 \\
\Rightarrow y &= (y-1) / (1-x)
\end{align*}
\]

Alternatively, the variable \( y \) could be solved for \( x \)

\[
\begin{align*}
(1/x) + (1/y) &= 1 \Rightarrow (1/y) = ((x-1)/x) \Rightarrow y = (x / (1-x))
\end{align*}
\]

\[
y = (((x-1) -x) / ((x-1) ²)) = - (1 / ((x-1) ²))
\]

the two results match.

Considering the two variables parameterized by the same variable \( t \), obtain a parametric equation

\[
E(x(t), y(t)) = c
\]

Applying differential operator \( D_{x} \) is obtained

\[
E'(x(t), y(t)) = 0
\]

Example: Given the equation \( x^2 + y^2 = 16 \) obtain a parametric equation and the relationship between \( x \) and \( y \) growth rates \( 2x \).

\[
\begin{align*}
x^2 (t) + y^2 (t) &= 0 \Rightarrow x' (t) = - (y')
\end{align*}
\]

This latter example is a type of problems is known as related fees \( y' \).
**Polynomial Approximation Function**

A function cannot be approximated only by a line but by a polynomial.

(Taylor) If a function fits derivatives exist and are continuous in \(a, b\), then in a neighborhood of \(x_0 \in [a, b]\), the function can be represented by a polynomial and rest.

\[ f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(\xi) \]

onde \(\xi \in V_{h}(x_0)\) \(h = x-x_0\) is a local variable.

The polynomial in the approach called Taylor polynomial of nth degree

\[ P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \]

The rest in the form of Lagrange

\[ R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \]

The special case where approximation is made in the vicinity of \(x = 0\) is called polynomial Maclaurin

\[ P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \]

Ex: Calculate the polynomial of degree Taylor 3, the function \(f(x) = e^x\), in the vicinity of \(x_0 = 0\). As \(f^{(n)}(x) = e^x\) and for any n, there has

\[ P_3(x) = e^0 + \frac{(e^0)}{1!}x + \frac{(e^0)}{2!}x^2 + \frac{(e^0)}{3!}x^3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \]

**Conclusion**


**Rating**

Group at least 5 and a maximum of 10 of these questions in each 1 hour evaluation test without reference.

Choose different type questions in the preparation of each test, changing the parameter and data base to avoid collages.

Each question is worth between 10% and 20% depending on the number of selected questions.

**Question 1:** Calculate the derivative of the function \(y = x^2 + 4\)

**Answer:**

\[ y' = 2x \]
Question 2: Calculate the derivative of the function \( y = 1 - x^3 \)

Answer:

\[ y' = -3x^2 \]

Question 3: Find the equation of the tangent line to the graph of the function at the given point: \( y = 2x - x^3 \), \( y(-2) = 4 \)

Answer:

\[ y'(x) = 2 - 3x^2 \]

\[ y(-2) = -10 \]

\[ y = 4 - 10(x + 2) = -10x - 16 \]

Question 4: Calculate the derivative of the function, applying the definition: \( y = 8 - 5x \)

Answer.

\[ \lim_{h \to 0} \frac{(8 - 5x - 5h) - (8 - 5x)}{h} = \lim_{h \to 0} (-5) = -5 \]

Question 5: Calculate the derivative of the function \( y = x^3 \)

Answer: \( \frac{d}{dx} (x^3) = 3x^2 \)

Question 6: Calculate the derivative of the function \( y = \sqrt{3x + 5} \)

Answer: \( D_x (\sqrt{3x + 5}) = \frac{3}{2\sqrt{3x + 5}} \)

Question 7: Calculate the derivative of the function:

Answer: \( f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \)

\[ f'(2) = \frac{-1}{5} \]

Question 8: Compute the derivative function by applying the definition: \( f(x) = (x + x^2 - 4) \)

Answer \( f'(x_0) = \lim_{x \to x_0} (f(x) - f(x_0))/(x - x_0) \)

\[ f'(4) = 7 \]
Question 9: Compute the derivative of the function \( y = \frac{1}{\sqrt{(x-1)}} \)

Response: \( y = -\frac{1}{2\sqrt{(1-x)(x-1)}} \)

Question 10: If \( f(a + b) = f(a) \cdot f(b),\) \( f(0) 1 = f'(0) \) exists for any \( a, b \in \text{dom}f \) then \( f'(x) = f'(0) \cdot f'(x) \)

Answer:

To prove that \( f(x) \) for all \( x \) exists note that any number can be written as \( x = 0x + \) then \( f(x + 0) = f(x) \cdot f(0) = f(x) = f\cdot 1(x) \)

To demonstrate that \( f'(x) = f'(0) \cdot f'(x) \)

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = f'(x) \cdot f'(0)
\]

Question 11: Make sure you have continuous and calculate the derivative of the left and right of the function at the point indicated to see if it is differentiable:

\( f(x) = \begin{cases} x^2 - 4 & \text{is } x < 2 \text{ and } \sqrt{(x-2)} \text{ was } 2 \leq x; x_0 = 2 \end{cases} \)

Answer:

\[
\lim_{x \to 2^-} f(x) = 0 = \lim_{x \to 2^+} f(x)
\]

\( f'(x) = \begin{cases} 2x & \text{if } x < 2, \text{ and } (1 / (2\sqrt{(x-2)})) \text{ is } 2 < x \text{ is not defined at } x = 2 \end{cases} \)

Question 12: Make sure you have continuous and calculate the derivative of the left and right of the function at the point indicated to see if it is differentiable: \( f(x) = (x-2)^{-2}; x_0 = 2 \)

Answer:

The function is not defined at \( x = 2 \)

\( f'(x) = -2 (x-2)^{-3} \)

The derivative function also is not defined at \( x = 2 \)

Question 13: Make sure you have continuous and calculate the derivative on the left and the right of the function at the point indicated to see if it is differentiable: \( f(x) = \begin{cases} x^2 & \text{if } x < 1, \text{ ax } + b \text{ is } x \geq 1 \end{cases} \)

Answer:

\[
\lim_{x \to 1^-} f(x) = 1 = a + \lim_{x \to 1^+} f(x)
\]

\( f'(x) = \begin{cases} 2x & \text{if } x < 1, \text{ and if } x \geq 1 \end{cases} \)

\( f_-'(1) = 2 = a = f_+'(1) \)

soon \( a = 2 \) and \( b = -1 \)

Question 14: Calculate the derivative of the function: \( g(x) = 8-3x \)

Answer: \( g'(x) = -3 \)
Question 15: Calculate the derivative of the function: \( g(x) = x^7 - 2x^5 + 5x^3 - 7x \)

Answer:
\[ g'(x) = 7x^6 - 10x^2 - 7 \]

Question 16: Calculate the derivative of the function \( f(x) = x^4 - 5 + x^{-2} - 4x^{-4} \)

Answer:
\[ f'(x) = 4x^3 - x^{-3} - 16x^{-5} \]

Question 17: Calculate the derivative of the function: \( g(x) = (2x^2 + 5)(4x - 1) \)

Answer:
\[ g'(x) = 4x(4x - 1) + 4(2x^2 + 5) = 4x + 24x^2 - 20 \]

Question 18: Compute the derivative of the function \( f(x) = (x^2 - 4 - 3x) / (x - 2) \)

Answer:
\[
\frac{d}{dx} \left( \frac{(x^2 - 4 - 3x)}{(x - 2)} \right) = \frac{((-2x - 3)(x - 2) - (4 - 3x - x^2))}{(x - 2)^2}
\]

Question 19: Calculate the derivative of the function \( f(s) = (s^2 - a^2) / (s^2 + a^2) \)

Answer:
\[
\frac{d}{ds} \left( \frac{(s^2 - a^2)}{(s^2 + a^2)} \right) = \frac{((-2s)(s^2 + a^2) - (2s)(s^2 - a^2))}{(s^2 + a^2)^2}
\]

Question 20: Calculate the equation of the tangent line to the graph of the function \( y = x^4 - 6x \), which is perpendicular to \( x - 2y + 6 = 0 \)

Answer:

The perpendicular to the \( x - 2y + 6 = 0 \) is a straight line with slope \(-1 / ((1/2)) = -2\)

What are the points where the line has this slope?

\[ y = 4x^3 - 6 = 1 = -2 \Rightarrow x \]

\[ y(1) = -5 \]

\[ y = -5 - 2(1 - x) = -2x - 3 \]

Question 21: Calculate the derivative of the function \( y = \sin x + \cos x \)

Answer: \((\sin x + \cos x)' = \cos x - \sin x\)

Question 22: Calculate the derivative of the function: \( g(y) = y - 3\sin y \cos y \)

Answer:
\[ g'(y) = y \cos y - y \sin y = 2 \cos y + y \sin y \]

Question 23: Calculate the derivative of the function \( f(t) = \sin t \cdot \tan t \)

Answer: \( f'(x) = \cos x + \sin \tan t \cdot \sec^2 t \)
Question 24: Calculate the derivative of the function \( y = (\sin t) / t \)

Answer: \( (d / (dt)) \cdot ((\sin t) / t) = ((t \cdot \sin t \cdot \cos t) / (t^2)) \)

Question 25: Calculate the derivative of the function: \( z = (\tan y + 1) / (\tan y - 1) \)

A. \( D_z \cdot y \cdot ((\tan y + 1) / (\tan y - 1)) = ((\sec^2 y \cdot (\tan y - 1) - (\tan y + 1) \sec^2 y) / ((\tan y - 1)^2)) = -2 \cdot \sec^2 y \)

Question 26: Calculate the derivative of the function at the given point: \( f(x) = x \cdot x^2 \cdot \cos \sin x \)

Answer:

\( f'(x) = (2x - 1) \cdot x \cdot \cos x \cdot x^2 \cdot \sin x \)

\( f'(0) = (0 - 1) = -1 \cdot \cos 0 - 0^2 \cdot \sin 0 \)

Question 27: Compute the derivative of the function at the point indicated: \( f(x) = (1 / (\cot x - 1)) \)

Answer:

\( f'(x) = (-\csc^2 x) / (\cot x - 1) \)

\( f'((3\pi) / 4) = (-\csc^2 (((3\pi) / 4))) / (\cot (((3\pi) / 4)) - 1)) = 1 \)

Question 28: Calculate the equation direct a tangent to the function graph in the indicated point: \( f(x) = x \cdot \sec \)

Answer:

\( f'(x) = x \cdot \tan x \cdot \sec \)

(A) \( x = \pi / 4 \)

\( f'((\pi) / 4)) = \tan ((\pi) / 4)) \cdot \sec ((\pi) / 4)) = \sqrt{2} \)

\( f ((\pi) / 4)) \cdot \sec = ((\pi) / 4)) = \sqrt{2} \)

\( y = \sqrt{2} + \sqrt{2} (x - (\pi) / 4)) \)

(B) \( x = -\pi / 4 \)

\( f'(- (\pi) / 4)) = \tan (- (\pi) / 4)) \cdot \sec (- (\pi) / 4)) = -\sqrt{2} \)

\( f (- (\pi) / 4)) \cdot \sec = (- (\pi) / 4)) = \sqrt{2} \)

\( y = \sqrt{2} + -\sqrt{2} (x + (\pi) / 4)) \)

(C) \( x = (3\pi) / 4 \)

\( f'(((3\pi) / 4)) = \tan (((3\pi) / 4)) \cdot \sec (((3\pi) / 4)) = \sqrt{2} \)

\( f (((3\pi) / 4)) \cdot \sec = (((3\pi) / 4)) = -\sqrt{2} \)

\( -\sqrt{2} + y = \sqrt{2} (x - ((3\pi) / 4)) \)

Question 29: Calculate the derivative of the function \( y = (x^3 - 3x^2 + 1)^{-3} \)

Answer: \( (d / (dx)) \cdot ((x^3 - 3x^2 + 1)^{-3} \cdot 3 \cdot (x - 2) / ((1 + (x^3 - 3x^2)^4)) \)}
Question 30: Calculate the derivative of the function \( y = (\sec^2 x) \)

Answer: \( \left( \frac{d}{dx} \right) (\sec^2 x) = (2 / (\cos^3 x)) \sin x \)

Question 31: Calculate the derivative of the function \( y = (4x^2 + 7)^2 (2x^3 + 1)^4 \)

Answer: \( \left( \frac{d}{dx} \right) (4x^2 + 7)^2 (2x^3 + 1)^4 = 8x (2x^3 + 1)^3 (64x^2 196x^3 + + + + + 8x^2 147x + 14) \)

Question 32: Calculate the derivative of the function \( y = (x + 3)^3 (5x + 1) (3x^2-4) \)

Answer: \( \left( \frac{d}{dx} \right) (x + 3)^3 (5x + 1) (3x^2-4) = 2 (x + 3)^2 (+ 45x^3 75x^2-31x-36) \)

Question 33: Calculate the derivative of the function \( y = (((\sqrt(x-1)) / (\sqrt(x+1)))) \)

Answer: \( \left( \frac{d}{dx} \right) (((\sqrt(x-1)) / (\sqrt(x+1)))) = (1 / (\sqrt(1-x) (x+1)^{3/2})) \)

Question 34: Calculate the derivative of the function \( y = (\sqrt(x^2 -5) [3] \sqrt(x^2 +3)) \)

Answer: \( \left( \frac{d}{dx} \right) (\sqrt(x^2 -5) [3] \sqrt(x^2 +3)) = (1/3) (x / (x^2 +3) \sqrt(x^2 -5)) [3] \sqrt(x^2 +3) (5x^2 -1) \)

Question 35: Calculate the derivative of the function \( y = (\sqrt((\cos x-1) / (\sin x))) \)

Answer: \( \left( \frac{d}{dx} \right) \sqrt((\cos x-1) / (\sin x)) = (1 / (2 (\sin x) \sqrt((1 / (\sin x)) (\cos x-1)))) \)

Question 36: Calculate the derivative of the function \( y = (\sqrt(xtan((1 / x))) \)

Answer: \( \left( \frac{d}{dx} \right) \sqrt(xtan((1 / x))) = -(1/(2\sqrt((1/x)))((1/(x^{3/2}))tan^2((1/x)+(1/(x^{3/2}))))-(1/(\sqrt(x))) \)

Question 37: Calculate the implicit derivative of the equation: \( x^2 + y^2 -7xy = 0 \)

Answer: \( 2x-7y-7xy' = 0 \Rightarrow y' = ((7y-2x) / (2y-7x)) \)

Question 38: Calculate the implicit derivative of the equation: \( (x / (\sqrt(y)) - 4y-x = 0 \)

Answer: \(-4y' + (1 / (\sqrt(y)) - (1/2) (x / ((y) ^ {3/2})))) = 1 Y' - 0 \Rightarrow y' = -2y ((\sqrt(y-1)) / (x + 8y ^ {3/2})) \)

Question 39: Calculate the derivative of the implicit equation: \( x - \sin(x + y) = 0 \)

Answer: \( 1- (\cos (x + y)) (y + 1) = 0 \Rightarrow y' = (1 / (\cos (x + y)))) - 1 \)

Question 40: Calculate the implicit derivative of the equation: \( \csc (xy) + \sec (x + y) = -x 0 \)

Answer: \( ((\cos (xy)) / (\sin^2 (xy))) y - ((\cos (xy)) / (\sin^2 (xy))) + (1 / (\cos^2 (x + y))) (\sin (x + y)) y + (1 / (\cos^2 (x + y))) (\sin (x + y)) - 1 = 0 \)

Answer: \( y = (((\cos (xy)) / (\sin^2 (xy))) + 1- (1 / (\cos^2 (x + y))) (\sin (x + y))) / ((1 / (\cos^2 (x + y))) (\sin (x + y)) ((\cos (xy)) / (\sin^2 (xy)))) \)
Question 41: Calculate the implicit derivative of the equation: \( y \sqrt{x} x \sqrt{y} = 9 \)

Answer:
\[
x' = \frac{\left( \frac{1}{2} \frac{jy}{jx} \right) x \sqrt{x} \left( \frac{1}{2} \frac{jx}{jy} \right) x - \frac{1}{2} \frac{y}{\sqrt{x}} x' - \frac{1}{2} \frac{x}{\sqrt{y}} y' = 0 \]
\]

Question 42: Show that the equation of the line tangent to \( x^2 + y^2 = r^2 \) is normal and passes through the center of the circumference.

Answer:
\[
2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}
\]

At the point \((x_1, y_1)\) has \( y' = -\frac{(x_1)}{(y_1)} \) is the slope of the tangent line

and the equation of the tangent line is \( y = y_1 - \frac{(x - x_1)}{(y_1)} x + \frac{y_1}{(y_1)} x_1 = 0 \)

The line passing through the center and the point \((x_1, y_1)\) of the circumference has slope \( \frac{(y_1)}{(x_1)} \)

Soon its equation is \( y = \frac{(y_1)}{(x_1)} x = xy_1 - xy_1 = 0 \)

Two lines \( Ax + By + C = 0 \) and \( A'x + C + B'y = 0 \) iff are perpendicular \( AA'BB' = 0 \) (or equivalently, if the slopes are related by the equation \( -\frac{A}{B} = -\frac{(A')}{(B')} \)) \(\Rightarrow\) \( \frac{A}{B} = \frac{(B')}{(A')} \) \(\Rightarrow AA' + BB' = 0 \)

Both conditions are met because:
\[-y_1x_1x_1y_1 + 0 = 0 \]

Pergunta 43: Calcule o polinómio de Taylor de n-ésimo grau com resto de Lagrange: \( f(x) = \tan x; x_0 = 0; n = 4 \)

Answer:
\[
\frac{d(f)}{dx}(\tan x) = \tan^2x+1; f'(0) = 1
\]
\[
\frac{d^2(f)}{dx^2}(\tan x) = 2(\tan x)(\tan^2x+1); f''(0) = 0
\]
\[
\frac{d^3(f)}{dx^3}(\tan x) = 6\tan^4x+8\tan^2x+2; f'''(0) = 2
\]
\[
\frac{d^4(f)}{dx^4}(\tan x) = 8(\tan x)(3\tan^4x+5\tan^2x+2); f^{(iv)}(0) = 0
\]
\[
\frac{d^5(f)}{dx^5}(\tan x) = 120\tan^6x+240\tan^4x+136\tan^2x+16; f^{(v)}(0) = 16
\]
\[
f(x) = x+(1/3)x^3+(120\tan^6x+240\tan^4x+136\tan^2x+16)/(720)x^6
\]

Question 44: Prove the following statements:

Answer:
\[
(X)' = 1
\]
\[
\lim_{h \to 0} \frac{(x + hx)}{h} = 1
\]
Question 45: Prove the following statements ::

Answer: \(( u^n \)' = nu^{n-1}u'\) where \(u(x)\) is a function of \(x\)

At issue 3 and chain rule

Question 46: Prove the following statements ::

Answer: \(( \sqrt{u} \)' = \( ( u' ) / ( 2\sqrt{u} ) \) for \(u(x) \geq 0\)

On one of the previous problems

\(( U^{(1/2)} \)' = (1/2) U^{-1/2}u'\)

= \( ( ( U' ) / ( 2\sqrt{u} ) )\)

Activity 3 - Extremes of functions in R

Introduction

Local and global extremes. Theorems of Fermat, Weierstrass (revisits), Cauchy, Rolle, Lagrange and l’Hospital. Monotony functions and concavity graphics. Tracing function graph. Differential of a function, the Leibniz notation. Rudiments of differential equations of 1st order.

Activity Details

Differentiability and local extremes

A function \(f\) is a local extremum (or on) a \(c \in \text{dom}f\) point if any \(V_\epsilon(c)\) neighborhood \(\{ x \in \epsilon(c) \} \) where \(f(c) \geq f(x)\), (local maximum), \(f(c) \leq f(x)\) (local minimum) to any \(x \in V_\epsilon(c)\).

A function \(f\) has a global extreme (or absolute) iff a range, there is some point in the interval \(C\) such that \(f(c) \geq f(x)\) (global maximum), \(f(c) \leq f(x)\) (min global) for any \(x\) in the interval.

Fermat’s theorem states that in extreme locations, the derivative, if it exists, is void:

( Fermat ) Let \(f\) be a function defined for all \(x \in \text{values}\) \(a, b\) [ with extreme location \(c \in ]a, b[\). If \(f'(c)\) then exist \(f'(c) = 0\).

The statement is made as follows: let \(c \in ]a, b[\) be a local maximum (mutatis mutandis to local minimum).

Then \(\exists \delta > 0\), \(|xc| < \delta \Rightarrow f(x) \leq f(c)\), then \(f(x) - f(c) \leq 0\)

For \(x \to c^+\) we have that \(0 < xc < \delta\) logo

\(\lim_{c^+} \{ x \to \((f(x) - f(c)) / (xc)\) \} \leq 0\)

For \(x \to c^-\) we have that \(0 < cx < \delta\) logo

\(\lim_{c^-} \{ x \to \((f(x) - f(c)) / (xc)\) \} \geq 0\)

It follows that \(f'(c)\) exist then the two limits laterals have to be equal, then \(\lim_{c^+} \{ x \to c \((f(x) - f(c)) / (xc)\) \} = 0\) and consequently \(f'(c) = 0\).
Note that the derivative may not exist at a point where the function $f$ has a local extreme [ see picture below ], (but if it exists, has necessarily to be zero at this point).

The reciprocal of Fermat’s theorem is not valid as shown by the following example:

Example:

The function $f(x) = (x - 1)$ has $f'(1) = 0$ but does not have a local extremum at $x = 1$

Example:

Determine the overall (or absolute) extreme of the function $f(x) = 2x$ in the range $[1, 4]$. The global minimum value of $f$ is in the interval $f(1) = 2$, and the function has a global maximum value for $\lim_{x \to 4} f(x) = f(8) < 8$ for $x \in [1, 4]$ but there is no point in the interval $[1, 4]$ where the function value is 8.

A critical point of the function $f$ is a $c \in \text{dom} f$ point such that $f'(c) = 0$ or $f'(c)$ does not exist.

Example:

Find the critical points of the function $g(x) = x^{(6/5)} - 12x^{(1/5)}$. Calculating the derivative $g'(x) = (6/5)x^{(1/5)} - (12/5)x^{(-4/5)} = 0 \Rightarrow x$
The point \( x = 0 \) pertende the domg but does not belong to domg ‘ , so the critical points are 0 and 2 .

By the Weierstrass theorem, if a function is continuous in the closed interval \([a, b]\) , then \( f \) has both global extremes that range.

To find the global extremes of a function is the set of function values at critical points and actual edges of the field . The maximum and minimum of this set are the popular values:

Ex: Find the absolute extremes of the function \( h (x) = (x - 3)^2 - 1 \) in the range \([2,5]\) . Since the function is continuous in the closed interval \([2,5]\) the theorem can be applied . The critical points are \( h ' (x) = 0 \Rightarrow 2 (x - 3) = 0 \Rightarrow x = 3 \) . The function values at critical points in the interval is \( f (3) = -1 \) and the edges of the interval is \( f (2) = 0 \) and \( f (5) = 3 \) . The extreme values are the function \( \max \{-1,0,3\} \) , and \( 3 = \min \{-1,0,3\} = -1 \), corresponding to the sections 5 and 3 respectively.

Theorems of Rolle, Lagrange, Cauchy and l'Hospital

(Rolle) If \( f \) is a continuous function at the interval \([a, b]\) and differentiable \([a, b]\) , such that \( f (a) = f (b) \), then there \( c \in ]a, b[ \) such that \( f '(c) = 0 \).

The statement is made as follows: if \( f \) is constant, any interior point of \([a, b]\) satisfies the proposition.

If \( f \) is not constant, the Weierstrass theorem, \( f \) has both global extremes on \([a, b]\) and at least one of these ends is obtained in a point inside \( c \in ]a, b[ \) where \( f (c) \neq f (a) = f (b) \), since \( f \) is not constant.

Since \( f \) is differentiable on \([a, b[ \), the derivative exists at the point \( c \), then by Fermat's theorem, \( f '(c) = 0 \)

Ex: The function \( f (x) = \sin 2x \) in the range \([0, \pi / 2]\) satisfies the Rolle theorem conditions for \( \sin 0 = 0, \sin 2(\pi / 2) = 0 \) and \( f (x) \) is differentiable in \([0, \pi / 2]\) [that is, \( f '(x) = 2\cos 2x \) exists in the interval] \( 0, \pi / 2 \]. Logo at some point \( 0, \pi / 2 \) [we have that \( f '(x) = 0 \). Solving equation \( 2\cos 2x = 0 \), we obtain \( 2x = (\pi / 2) \Rightarrow x = (\pi / 4) \).

(Lagrange or average value) If \( f \) is a continuous function in the interval \([a, b]\) and differentiable on \([a, b[ \), then there \( c \in [a, b[ \) such that

\[
 f '(c) = \left( \frac{f(b) - f(a)}{b-a} \right)
\]

The statement is made as follows: Consider the auxiliary function

\[
 h (x) = [BA] f (x) - [f (b) - f (a)] x
\]

It is noted that \( H (A) = h (b) = bf (a) - AF (b) \) and \( h (x) \) satisfies the conditions of Rolle theorem, then there \( c \in ]a, b[ \) such that \( h '(c) = 0 \).

As

\[
 h '(x) = f ' (x) - \frac{f (b) - f (a)}{(b-a)}
\]

So

\[
 f '(c) - \frac{(f(b)-f(a))/(b-a)}=0
\]
Example:

The function $f(x) = x^{(2/3)}$ in the range $[0,1]$ satisfies the Lagrange theorem conditions: it is continuous on $[0,1]$ and differentiable in $[0.1]$, namely, $f'(x) = (2/3)x^{(-1/3)}$ is set to $[0.1]$. It has to be

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Then there is a number $c \in [a, b]$ such that $f'(c) = 1$

$$\frac{(2/3)c^{(1/3)}}{1} = c = ((2/3)^3$$

(Cauchy) If $f$, $g$ are continuous in the interval $[a, b]$, differentiable on $[a, b]$, then $g(a) \neq g(b)$ and there $c \in (a, b)$ such that

$$\frac{(f(b) - f(a))}{(g(b) - g(a))} = \frac{(f'(c))}{(g'(c))}$$

The statement is made as follows: if $g(a) = g(b)$ then by Rolle $g$ theorem ‘to cancel at some point’ $a$, $b$ [i.e., contrary to the hypothesis, then $g(a) \neq g(b)$).

Consider the auxiliary function

$$h(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x)$$

It is noted that $H(a) = h(b) = g(b)f(a) - f(b)g(a)$ satisfies the conditions of Rolle theorem, then there $c \in (a, b)$ such that $h'(c) = 0$.

So

$$h'(c) = [g(b) - g(a)]f'(c) - [f(b) - f(a)]g'(c) = 0$$

(L'Hospital (0/0)) where $f$, $g$ are differentiable in $[a, b]$, ‘not canceled in’ $a$, $b$ [i.e., $\lim_{x \to a^+} f(x) = \lim_{x \to b^-} g(x) = 0$, then $x \to a^+ \lim_{x \to a^+} (f(x)) / (g(x)) = (x \to b^- \lim_{x \to b^-} ((f'(x)) / (g'(x))), if any.

The statement is made as follows: as $\lim_{x \to a^+} f(x) = \lim_{x \to b^-} g(x) = 0$ can be extended $f$, $g$ by continuity and get $f(a) = f(b) = 0$.

Then $f$, $g$ will be continuous on the interval $[a, b]$.

If $x \in E$, $b$ [can apply the Cauchy’s theorem $f$, $g$ in the interval $[a, x]$ and concluding that there $y \in E$ to $x$ such that

$$\frac{(f(x))}{(g(x))} = \frac{(f'(y))}{(g'(y))}$$

As $y \to a^+$ when $x \to a^+$, is obtained

$$\lim_{x \to a^+} (f(x)) / (g(x)) = \lim_{x \to a^+} (f'(x)) / (g'(x))$$

Note: The theorem also applies (mutatis mutandis) to $x \to b^-$ for $x \to c \in E$ $a$, $b$ [for $x \to + \infty$ or $x \to -\infty$.}
As a corollary, if \( f \) and \( g \) are differentiable functions in \( a, + \infty \), where \( a > 0 \), \( g \) ‘is not canceled in\( a, + \infty \), \( \lim_{x \to + \infty} f(x) = \lim_{x \to + \infty} g(x) = 0 \), then \( \lim_{x \to + \infty} \left( \frac{f(x)}{g(x)} \right) = \lim_{x \to + \infty} \left( \frac{f'(x)}{g'(x)} \right) \), if any.

(L'Hospital \((+ \infty) / (+ \infty)\)) If \( f, g \) are differentiable on \( a, b \) \( g \) ‘is not canceled in\( a, b \), \( \lim_{x \to a^+} f(x) = \lim_{x \to b^-} g(x) = + \infty \), then \( \lim_{x \to a^+} \left( \frac{f(x)}{g(x)} \right) = \lim_{x \to b^-} \left( \frac{f'(x)}{g'(x)} \right) \), if any.

Note: The theorem also applies (mutatis mutandis) to \( x \to b^- \) for \( x \to c \in a, b \) \( [x \to + \infty \) or \( x \to -\infty \).

Monotony functions and concavity graphics

(Monotone) Let \( f \) be a continuous function in an interval \( I \) and differentiable in the interior of \( I \). If for any \( x \in \text{int}(I) \):

\[
f'(x) \geq 0, \text{ then } f \text{ is increasing on } I;
\]
\[
f'(x) > 0, \text{ then } f \text{ is strictly increasing on } R;
\]
\[
f'(x) \leq 0, \text{ then } f \text{ is decreasing in } R;
\]
\[
f'(x) < 0, \text{ then } f \text{ is strictly decreasing in } R;
\]
\[
f'(x) = 0, \text{ then } f \text{ is constant};
\]

The statement is made as follows: in the first case, \( f'(x) \geq 0 \) for any \( x \in \text{int}(I) \).

Let, \( b \in I \) such that \( a < b \), or \( ba > 0 \)

The mean value theorem applied to the restriction \( f \mid_{[a, b]} \) there \( c \in a, b \) [such that

\[
(f(b) - f(a)) / (b-a) = f'(c) \geq 0
\]

As \( ba > 0 \), it follows that \( f(b) - f(a) \geq 0 \), ie \( f(b) \geq f(a) \), then \( f \) is increasing.

Other cases are shown similarly.

The value of the derivative of a continuous function can only change signal at critical points.

At a critical point \( c \), where \( f'(c) \) does not exist or \( f'(c) = 0 \), can be done the first derivative test to determine whether a maximum or a local minimum:

Test first derivative for local extremes

if \( f'(x) > 0 \) for all values of some interval \( c-\delta, c \) \( f'(x) < 0 \) for all values of some interval \( c, c + \delta \), then \( f \) have a local maximum \( c \);

if \( f'(x) < 0 \) for all values of some interval \( c-\delta, c \) \( f'(x) > 0 \) for all values of some interval \( c, c + \delta \), then \( f \) have a local minimum \( c \);

Example: Given the function \( f(x) = x^3 - 6x^2 + 9x + 1 \), the derivative exists for all \( R \). The critical points are obtained by

\[
f'(x) = 12x + 9 - 3x^2
\]
\[
3 = (3-x)(x-1) = 0
\]
Logo $x = 3v_1x = 1$ are the critical points

<table>
<thead>
<tr>
<th>$X$</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>crescete</td>
<td>max. loc.</td>
</tr>
</tbody>
</table>

The sign of the second derivative reported on the graph concavity: the graph of $f$ differentiable function in some open interval containing $c$ is concave upward at $(c, f(c))$ if $f''(x) > 0$ and is concave down $(c, f(c))$ if $f''(x) < 0$.

The reciprocal is not true, for example, point $(0,0)$ function graph $x = x^4$ is concave upwards but $f''(x) = 0$.

The point $(c, f(c))$ is a point of inflexion of the graph if there is a tangent line at that point and a neighborhood $V_\left\{ \delta \right\}(c)$ such that $x \in V_\left\{ \delta \right\}(c)$ so:

$f'(x) < 0$ if $x < c$ and $f'(x) > 0$ if $x > c$ or

$f'(x) > 0$ if $x < c$ and $f'(x) < 0$ if $x > c$ or

If the function $f$ is differentiable $[a, b]$, $c \in [a, b]$, and $(c, f(c))$ is an inflexion point of the graph of $f$ if $f'(x)$ then there is $f'(x) = 0$.

The statement is made by applying Fermat's theorem $f'(x)$.

The reciprocal is not true as you can see in paragraph $(0.0)$ of the function $f$ graph $(x) = x^4$.

Example:

The example inflection point is obtained

$f''(x) = 6x - 12 = 0$
Then the inflection point is (2,3) . The result is summarized in the table

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>f'(x)</th>
<th>f''(x)</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>+</td>
<td>+</td>
<td>cres.</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>máx. loc.</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-3</td>
<td>-0</td>
<td>decres.</td>
</tr>
</tbody>
</table>

Testing of the second derivative for local extremes: Let f’(c) = 0 and f’(x) exists in a neighborhood of c . If f’(x) exist and

- f''(c) > 0, then f has a local minimum in C
- f''(c) < 0 then f has a local maximum in C

Trace function graph
The procedure for a function chart of the study is the following:

1. Determine the domain and the edges of the domain of f
2. Find symmetries and translations
3. Find the zeros and f signal between zeros or lips
4. Find the asymptotes (limits on domain lips)
5. Calculate the first derivative f’
6. Find the critical points
7. Study the monotony of fa from the signal f’
8. Calculate the second derivative f’’
9. Find the inflection points
10. Study the bending fa from the f signal ‘’

Differential of a function, the Leibniz notation
The increments of the variables x and y are represented by Δx and Δy respectively.
Given a function \( y = f(x) \), the difference \( y \) is defined by
\[
\Delta y = f'(x) \Delta x
\]
The differential of \( x \) is given by
\[
\Delta x = (x)' \Delta x = x \\
\]
Of the \( \Delta x \) and \( \Delta y \) settings can get the relation between them when \( f'(x) \) exists
\[
\Delta y = f'(x) \Delta x
\]
The differential \( \Delta y \) is a linear function of \( \Delta x \). In Leibniz notation we have that
\[
f'(x) = \frac{\Delta y}{\Delta x}
\]
The chain rule for a composite function \( y = f(x) \) where \( x = g(t) \) is expressed by
\[
\frac{\Delta y}{\Delta t} = \left( \frac{\Delta y}{\Delta x} \right) \left( \frac{\Delta x}{\Delta t} \right)
\]
And the relationship between two variables differential depends on the functions that associate and is given by
\[
\Delta y = \left( \frac{\Delta y}{\Delta x} \right) \left( \frac{\Delta x}{\Delta t} \right) \Delta t
\]
The exterior derivative of a function \( f(x) \) defined in a related open \( R \) is given by
\[
\Delta f = f'(x) \Delta x
\]
The relationship between the exterior derivative and the derivative of a function is given by Leibniz notation
\[
\frac{\Delta f}{\Delta x} = f'(x)
\]
A differential form defined in a related open \( R \), is an expression of the type : \( g(x) \Delta x \)
Ahead will be deepened the properties of differential forms.

Rudiments of differential equations of 1st order

Sometimes we have a relationship between two variables differences and we want to determine the relationship between the two variables.
A differential equation of 1st order involves independent variables \( t, x \) dependent on its derivatives, can be written in explicit

\[
\frac{dx}{dt} = f(t, x)
\]

or implicitly

\[
F(t, x, \frac{dx}{dt}) = 0
\]

A solution of an equation in a \( \mathbb{I} \subset \mathbb{R} \) validity domain is an integral curve \( x(t) \) defined in an interval \( \mathbb{I} \), whose derivative is also defined in \( \mathbb{I} \) and that satisfies the given equation.

The general solution of an equation in a \( \mathbb{I} \subset \mathbb{R} \) validity domain is a family of solutions depending on an arbitrary parameter integration \( c \).

The arbitrary parameter \( c \) can be determined when a solution known value at a point \( x(t_0) = x_0 \), termed initial condition of the integral curve.

An initial value problem (PVI) is composed of an equation and an initial condition

\[
\frac{dx}{dt} = f(t, x)
\]

\[x(t_0) = x_0\]

A solution of PVI is an integral curve \( x(t) \) that satisfies a respective differential equation maximal interval \( \mathbb{I} \), such that \( t \in \mathbb{I} \) and whose derivative \( \frac{dx}{dt} \) is also defined in \( \mathbb{I} \).

The function \( f(t, x) \) of explicitly defining at each point \( (t, x) \) the slope of the tangent to the integral curve \( x(t) \), determining a direction field in the \( \mathbb{I} \times \mathbb{R} \) plane.

Ex: The family of solutions of the equation \( \frac{dx}{dt} = t - (\frac{2}{t}) \) is given by \( x(t) = (\frac{t^2}{4}) + (\frac{c}{(t^2)}) \).

They are represented some integral curves in the Figure below

![Integral curves of \( (dx)/(dt)=t-(2x/t) \)](integral_curves_of_((dx)/(dt))=t-(2x/t))

Some qualitative information about the solution may be withdrawn from the general form of the equation \( \frac{dx}{dt} = t\ (2x/t) \).

1. The validity domain is disjointed, solutions for \( t < 0 \) or \( t > 0 \);

2. The symmetry of the tangent straight inclination to \( t = 0 \)

\[
(\frac{Dx}{dt}) (t) = t - (\frac{2}{t}) = -((t - T) - ((2) / (t - T))) = ((dx)/(dt)) (-t)
\]
requires that the solution is an even function \( x(t) = x(-t) \);

3. The stationary points \( \left( \frac{dx}{dt} \right) = 0 \) are on the parabola \( x = \left( \frac{t^2}{2} \right) \) (marked by broken lines in the figure), which separates the regions where the slopes of the tangent lines are positive \( x < \left( \frac{t^2}{2} \right) \) of the region where the inclination is negative \( x > \left( \frac{t^2}{2} \right) \)

**Conclusion**

Local and global extremes of real functions were handled, presented the theorems of Fermat, Weierstrass (revisits), Cauchy, Rolle, Lagrange and l’Hospital. It was related to the monotony of functions and concavity graphics with the first and second derived deductions.

Respectively, the necessary and sufficient elements for the full stroke function graph were presented. Defined increment and differential a function, Presented the rating of Leibnitz and set the exterior derivative of a function. They were presented the rudiments of differential equations of 1st order.

**Formative Evaluation**

Group a minimum of 5 and a maximum of 10 questions in each of the following 1 hour evaluation test without reference.

Choose different type questions in the preparation of each test, the changing parameters and database to prevent bonding.

Each question is worth between 10% and 20% depending on the number of selected questions.

**Question 1:** Find the critical points of the function \( f(x) = \frac{(x + 1)}{(x^2-5x + 4)} \)

**Answer:**

\( \text{dom}_f = \{ x \in \mathbb{R} : x^2-5x + 4 = 0 \} = \{1,4\} \mathbb{R}^- \)

\( f'(x) = \frac{\left( ((x-1)(x-2) - (x + 1))((x-1) + (x-2)) \right)}{((x-1)^2(x-2)^2)} = -\frac{((x^2 + 2x-5))}{((x-1)^2(x-2)^2)} \)

\( f'(x) = 2x-0 \rightarrow x^2 + 5 = 0 \Rightarrow x = -1 = -1 + \sqrt{6} \)

**Question 2:** Minimize the function \( f(x) = F(\theta) = \frac{(m(a + kg))}{(\cos\theta + k\sin\theta)} \)

**Answer:**

\( F(\theta) = \frac{(m(a + kg))}{(\cos\theta + k\sin\theta)} \)

If the body moves at a constant velocity, then 0 =

\( F'(\theta) = \frac{(\cdot KG (-\sin\theta k\cos\theta +\cdot))}{((\cos\theta + k\sin\theta)^2)} = \frac{((KG (-\tan\theta + k))}{(\cos\theta (1 + k\tan\theta)^2)} \)

\( F'(\theta) = 0 \Rightarrow \frac{(KG (-\tan\theta + k))}{(\cos\theta (1 + k\tan\theta)^2)} = 0 \Rightarrow \)

\( KG (-\tan\theta + k) = 0 \Rightarrow \)
Question 3: Prove Rolle’s theorem to the function \( f(x) = x^{(3/4)} - 2x^{(1/4)} \) in the range \([0,4]\)

Answer:

\[
\text{dom} f = [0, + \infty [
\]

\( f \) is continuous on \([0,4]\) (rational exponents with positive square root)

\( f \) is differentiable in \([0,4]\)

\[
f(0) = 0 = f(4) = \sqrt[4]{2^6} - 2\sqrt[4]{2} = 2 \sqrt[2]{2^2} - 2\sqrt[4]{2} = 0
\]

Then there \( c \in ]0,4[ \) such that \( f'(c) = 0 \)

\[
f'(x) = \frac{3}{4} x^{-\frac{1}{4}} - \frac{1}{2} x^{-\frac{3}{4}}
\]

\[
f'(c) = 0 \Rightarrow \frac{3}{4}c^{-\frac{1}{4}} - \frac{1}{2}c^{-\frac{3}{4}} = 0 \Rightarrow \frac{3}{4}c^{-\frac{1}{2}} - \frac{1}{2} = 0 \Rightarrow c^{\frac{1}{2}} = \frac{2}{3} \Rightarrow c = \sqrt{\frac{2}{3}}
\]

Question 4: Find the extreme and sketch the function graph \( f(x) = x^5 - 5x^3 - 20x - 2 \)

Answer:

\[
f'(x) = 5x^4 - 15x^2 - 20 = 0
\]

\[
(X - 2)(x + 2)(x^2 + 1) = 0
\]

\(-2 \times 2
\]

\[
(X - 2) - - - + 0
\]

\[
(X + 2) - 0 + + +
\]

\[
(X - 2)(x + 2)(x^2 + 1) + 0 - 0 +
\]

The first derivative is zero at -2.2

At -2, the signal is positive and negative before then, so it is a local maximum

2, the signal is negative before and positive after, so it is a local minimum

\( x^5 - 5x^3 - 20x - 2 \)
Question 5: Find the extreme and sketch the graph of the function $f(x) = x^4 - 2x^3$

Answer:

$$f'(x) = 4x^3 - 6x^2$$
$$f''(x) = 12x - 12x^2 \Rightarrow x = 0, 1$$

Are the inflection points

$x - - - 0 +$
$(X + 1) - 0 + + +$
$x(x + 1) + 0-0 +$

Concave down between 0 and 1 and dished up in other regions.

Summary

In this unit showed up the continuity concepts derived functions in the geometric INTERPRETATION derived , the derived differential outside and forms functions .

The continuity of real functions of real variable were presented some local properties of continuous functions . Alternatively continuity characterization . Notable examples of continuous functions . Discontinuity functions. global continuous functions properties (Theorem of Bolzano or intermediate value )


As for local and global extremes seen the theorems of Fermat , Weierstrass ( revisits ) , Cauchy , Rolle, Lagrange and l' Hospital . Monotony functions and concavity graphics. Tracing function graph . Differential of a function , the Leibniz notation . Rudiments of differential equations of 1st order .

Unit Evaluation

Instructions

Group a minimum of 5 and a maximum of 10 questions in each of the following evaluation test of 1 hour without consultation .

Choose different type questions in the preparation of each test , the changing parameters and database to prevent bonding .
Rating criteria

Each question is worth between 10% and 20% depending on the number of selected questions.

Summative Evaluation

The following questions-type, with their detailed solutions:

Unit 1

Question 1: Determine discontinuities function $H(x) = \frac{1}{x + 2}$

Answer: Discontinuity at $x = -2$ because $\lim_{x \to -2} H(x)$ does not exist. ($\lim_{x \to -2^-} H(x) = -\infty$ and $\lim_{x \to -2^+} H(x) = +\infty$)

Question 2: $g(x) = \frac{(|x|)}{x}$ if $x \neq 0$, and 1 if $x = 0$

Answer: Discontinuous at $x = 0$ for $\lim_{x \to 0^-} \frac{(|x|)}{x} = -1$ and $\lim_{x \to 0^+} \frac{(|x|)}{x} = 1$

Question 3: Determine discontinuities function: $G(x) = \frac{(x-2)}{(x^2 + 2x-8)}$

Answer: It is discontinuous at $x = 2$ and $x = -4$ it is not set these points and is continuous at all other points.

Question 4: Determine discontinuities function $f(x) = \frac{\sqrt{2 + \sqrt{[3]x}} - 2}{(x-8)}$

Answer: $x = 8$ is discontinuous because it is not defined.

The component $[3] \sqrt{x}$ is defined but any $x \in \mathbb{R}$ $\sqrt{2 + [3] \sqrt{x}}$ delimits the domain of $f$ to $[3] \sqrt{x}$ $(x) \geq -8 - 2 \Rightarrow x \geq 8$.

Therefore dom$f = [-8, 8] U [8, +\infty]$.

Now, it is recalled the expression $ab = (a \wedge (1/n)) - b \wedge (1/n)) \Sigma_{k=1}^{n} a \wedge ((k-1)/n) \{b \wedge ((k-1)/n)\}$

Applied to the case it has

$X-8 = ([3] \sqrt{x-2}) ([3] \sqrt{x} + 2 [3] \sqrt{x} + 4)$

Then, it follows that

$\lim_{x \to 8} \frac{\sqrt{2 + \sqrt{[3]x}} - 2}{(x-8)} = \lim_{x \to 8}\frac{\{(\sqrt{2 + [3] \sqrt{x}}) - 2\}}{((x-8))} / ((3)\sqrt{(x-2)}$ $([3] \sqrt{(x^2) + 2[3] \sqrt{x} + 4})) / ((x-8) + 2[3] \sqrt{x} + 4)) / ((x-8) + 2[3] \sqrt{x} + 4))$  

$\lim_{x \to 8} = (8) / ((3) \sqrt{x-2}) / ((3) \sqrt{x-2}) / ((3) \sqrt{x+2}) / (2 \sqrt{(x-2)} + 2[3] \sqrt{x+4}) (1 / ((\sqrt{(2 + [3] \sqrt{x} + 2)}))$  

$= (1 / (48))$

Therefore, by setting $f(8) = (1 / (48))$, the function becomes continuous at $x = 8$. 
Question 5: Determine function of discontinuities:

\[ g(x) = \begin{cases} -1 & \text{if } x \geq 1 \\ 1 & \text{if } x < 1 \end{cases} \]

Answer: it is discontinuous at \( x = 1 \)

Question 6: Determine continuity of the composite function \( f \circ g \)

Answer:

\[ f(x) = \sqrt{x}, \quad g(x) = x^2 - 16 \]

\[(f \circ g)(x) = \sqrt{16 - x^2} \]

\( \text{gift } (f \circ g) = \{ x \in \mathbb{R} \mid 16 - x^2 \geq 0 \} = [-4, 4] \)

The function is continuous on its domain

Question 7: Determine continuity of the composite function \( f \circ g \)

Answer:

\[ f(x) = x^2; \quad g(x) = x^2 - 3 \]

\[(f \circ g)(x) = (x^2 - 3)^2 \]

\( \text{gift } (f \circ g) = \mathbb{R} \)

The function is continuous on its domain

Question 8: Determine continuity of the composite function \( f \circ g \)

Answer:

\[ f(x) = \left( \frac{1}{\sqrt{1-x^2}} \right) / \left( \sqrt{4-|x|} \right), \quad g(x) = |x| \]

\[(f \circ g)(x) = \left( \frac{1}{\sqrt{1-x^2}} \right) / \left( \sqrt{4-|x|} \right) \]

\( \text{gift } (f \circ g) = \{ x \in \mathbb{R} : 4|x| > 0 \land |x| \geq 1 \}\]

\( = \{ x \in \mathbb{R} : |x| < 4 \land |x| \geq 1 \}\]

\( = \{ x \in \mathbb{R} : -4 < x < 4 \land -1 \leq x < 1 \}\]

\( = [-4, -1) \cup [1, 4] \)

The function is continuous on its domain

Question 9: Determine continuity of the function \( f(x) = [x] \), remember \([x] = n \) if \( n \leq x < n + 1 \) where \( n \) is an integer

Answer:

\[ f(x) = [x], \text{ remember } [x] = n \text{ if } n \leq x < n + 1 \text{ where } n \text{ is an integer} \]

\( \text{dom } f = \mathbb{R} \)
On \((1/2)\), \((1/2)\) [the function is not continuous at 0]

At \((1/4)\), \((1/2)\) [the function is continuous]

In \([1, 2)\) [function is continuous]

In \([1, 2]\) [the function is continuous]

In \([1, 2]\) the function is discontinuous at 2

**Question 10:** Determine continuity of the composite function \(f \circ g\)

**Answer:**

\[ f(x) = \sqrt{x}; \quad g(x) = 16 + x^2 \]

\[(f \circ g)(x) = \sqrt{16 + x^2} \]

dom\((f \circ g) = \{x \in \mathbb{R} : 16 + x^2 \geq 0\} = \mathbb{R} \]

A função é contínua no seu domínio

**Question 11:** Determine continuity of the composite function \(f \circ g\)

**Answer:**

\[ f(x) = \left(\frac{\sqrt{4-x^2}}{\sqrt{x-1}}\right) ; \quad g(x) = |x| \]

\[(f \circ g)(x) = \left(\frac{\sqrt{4-|x|^2}}{\sqrt{|x|-1}}\right) \]

dom\((f \circ g) = \{x \in \mathbb{R} : |x| - 1 > 0 \land 4 - |x|^2 \geq 0\} \]

= \{ x \in \mathbb{R} : |x| > 1 \land |x|^2 \leq 4 \}

= \{ x \in \mathbb{R} : -2 \leq x \leq 2 \}

= \{-2, -1 \cup 1, 2\}

The function is continuous on \([-2, -1 \cup 1, 2]\)

**Question 12:** \(f(x) = 4x - x^3 + x + 3\) is continuous in the interval \([1, 2]\)

**Answer:**

The values of \(f(1) = 1\) \(f(2) = 5\)

The theorem of Bolzano only ensures that there is a \(c\) value \([1, 2]\) where \(1 \leq f(c) \leq 5\) (not a root, so the question is not correct)

**Unity 2**

**Question 1:** Calculate the derivative of the function \(y = x^2 - 6x + 9\)

**Answer:** \(y = 2x - 6\)
Question 2: Find the equation of the tangent line to the graph of the function at the given point: \( y = x^2 + 2x + 1 \), \( y(1) = 4 \)

Answer:

\[
y' = 2x + 2
y(1) = 4
= 4y + (x-1) = 4x \Rightarrow y
\]

Question 3: Find the equation of the tangent line to the graph of the function at the given point: \( y = -\frac{8}{(\sqrt x)} \), \( y(4) = -4 \)

Answer:

\[
y' = 2 - 3x^2
y'(-2) = -10
y = 4 - 10(x + 2) = -10x \Rightarrow y - 16
\]

Question 4: Calculate the derivative of the function, applying the definition:

\( y = 3x^2 + 4 \)

Answer:

\[
\lim_{h \to 0} \frac{((3(x + h)^2 + 4) - (3x^2 + 4))}{h} = \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 + 4) - (3x^2 + 4)}{h}
\]

\[
\lim_{h \to 0} (6x + 3h) = 6x
\]

Question 5: Calculate the derivative of the function \( y = \frac{1}{x + 1} \)

A. \( D_x \left(\frac{1}{(1 + x)}\right) = -\frac{1}{(1 + x)^2} \)

Question 6: Calculate the derivative of the function \( f(x) = x^3 \)

Answer:

\( f'(x) = 3x^2 \)

Question 7: Compute the derivative of the function by applying the definition: \( f(x) = \frac{2}{(\sqrt{x})} - 1 \)

A. \( \lim_{h \to 0} \frac{(2/ \sqrt{(x + h)}) - 1}{h} = \lim_{h \to 0} \frac{((2/ \sqrt{(x + h)}) - 1) / h}{((2/ \sqrt{x}) - 1) / h} = \frac{2/ \sqrt{(x + h)} - 2/ \sqrt{x}}{(h/ \sqrt{(x + h)}) - (h/ \sqrt{x})}
\]

\[
= \left(\frac{-1}{x \sqrt{x}}\right) \quad f'(x) = -\frac{1}{8}
\]
Question 8: Calculate the derivative of the function \( y = \frac{4}{2x-5} \)

Answer:

\[ y = \left(\frac{-8}{(2x-5)^2}\right) \]

Question 9: Prove that there is no tangent line to \( y = 4-x^2 \) and passing through the point \( y(1) = 3 \)

Answer:

The tangent line through the point \((1,3)\) immediately goes also for \((1.2)\) would be the line \( x = 1 \)

But the tangent to the curve at \( x = 1 \) is given by

\[ y' = -2x \]
\[ y(1) = -2 \]
\[ y = 3 - 2(x-1) = 5 - 2x \]

Question 10: Make sure you have continuous and calculate the derivative of the left and right of the function at the point indicated to see if it is differentiable: \( f(x) = 1 + |x+2|; x_0 = -2 \)

Answer:

\[ \lim_{x \to -2} (1+|x+2|) = 1 = f(-2) \] then the function is continuous

\[ f'_-(x) = \lim_{x \to -2^-} \frac{(f(x)-f(-2))/(x-(-2))}{(1+|x+2|)-(1+|-2+2|)/(x-(-2))} \]
\[ = \lim_{x \to -2^-} \frac{(-(x+2))/(x+2)}{1} \]
\[ = -1 \]

\[ f'_+(x) = \lim_{x \to -2^+} \frac{(f(x)-f(-2))/(x-(-2))}{(1+|x+2|)-(1+|-2+2|)/(x-(-2))} \]
\[ = \lim_{x \to -2^+} \frac{(x+2))/(x+2)}{1} \]
\[ = 1 \]

\( f \) is not differentiable at \( x = -2 \)

Question 11: Make sure you have continuous and calculate the derivative of the left and right of the function at the point indicated to see if it is differentiable:

\( f(x) = \begin{cases} x^2 & \text{if } x < 2, \\ -1-2x & 2 \leq x = -1 \end{cases} \)

Answer:

\[ \lim_{x \to -1^-} (x+\lim_{x \to -1} f(x)) = 1 = \lim_{x \to -1^+} f(x) \]

\( f'(x) = \begin{cases} 2x & \text{if } x < 2, \\ -2 \text{ to } 2 \leq x \end{cases} \)

\[ f'_-(-1) = \lim_{x \to -1^-} \frac{(1-x^2)}{(x - (-1))} \]
\[
\lim_{x \to -1^-} \frac{(x + 1)(x-1)}{x + 1} = -2
\]

\[f'(x) = \begin{cases} 2x & \text{if } x < -1 \land x > 1, \\ 2x & \text{if } -1 \leq x \leq 1 \end{cases}\]

\[f_+(-1) = -2 \neq f_-(-1) \text{ is not derivable}\]

Question 12: Make sure you have continuous and calculate the derivative of the left and right of the function at the point indicated to see if it is differentiable: \(f(x) = |1-x^2|\)

Answer:
\[\lim_{x \to -1^-} f(x) = 0 = \lim_{x \to -1^+} f(x) \text{ is continuous}\]

\[f_-'(x) = \begin{cases} 2x & \text{if } x < -1 \land x > 1, \\ 2x & \text{if } -1 \leq x \leq 1 \end{cases}\]

"The theorem states that if \(f\) is differentiable then is continuous"

Soon it is not continuous may not be differentiable

If \(x_1\) for all \(f_-'(x_1) = -\infty \text{ ef } f_+'(x_1) = [x_1] \text{ (check)}\)

Question 13: Calculate the derivative of the function when \(x_1\) is full and when it is not: \(f(x) = [x]\)

Answer:
If \(x_1\) is not all \(f_-'(x_1) = 0\)

The theorem states that if \(f\) is differentiable then is continuous

Soon it is not continuous may not be differentiable

If \(x_1\) for all \(f_-'(x_1) = -\infty \text{ ef } f_+'(x_1) = [x_1] \text{ (check)}\)

Question 14: Calculate the derivative of the function \(f(x) = 3x^4 - 5x^2 + 1\)

Answer:
\[f_-'(x) = 10x - 12x^3\]

Question 15: Calculate the derivative of the function: \(G(y) = y^{10} + 7y^5y^3 + 1\)

Answer:
\[G_-'(y) = 10y^9 + 35y^4 - 3y^2\]

Question 16: Calculate the derivative of the function \(H(x) = (5 / (6x^5)) = (5/6)x^{-5}\)

Answer:
\[H_-'(x) = \frac{(-150x^4)}{(36x^6)} = \frac{-25}{(6x^6)}\]

Question 17: Calculate the derivative of the function: \(F(t) = (t^3 - 2t + 1)(2t^2 + 3t)\)

Answer:
\[F_-'(t) = (3t - 2)(2t^2 + 3t) + (t^3 + 2t - 1)(4t + 3) + = 4t^4 + 9t^3 + 3t^2 - 8t + 3t + 2\]
Question 18: Calculate the derivative of the function \( f(x) = \frac{x^4 - 2x^2 + 5x + 1}{x^4} \)

Answer: \( \frac{d}{dx} \left( \frac{x^4 - 2x^2 + 5x + 1}{x^4} \right) = \frac{(4x^3 - 4x + 5)(x^4) - (x^4 - 2x^2 + 5x + 1)(x^4)}{x^8} \)

\( = \frac{1}{x^4} (4x^3 - 4x + 5 - 2x^6 + 9x^4) \)

Question 19: Calculate the equation of the line tangent to the graph of the function at the given point: \( y = \frac{8}{x^2 + 4} \), \( y(2) = 1 \)

Answer:

\( y' = -\frac{16x}{(x^2 + 4)^2} \)

\( y'(2) = -\frac{1}{2} \)

\( y = \frac{1}{2} (2-x) \quad 2 = \frac{1}{2} x + 1 \)

Question 20: Calculate the equation of the line tangent to the graph of the function \( y = \frac{3}{x^3} - x^2 + 2x + \frac{4}{3} \), parallel to \( 2x - y + 3 = 0 \)

Answer:

The line parallel to \( 2x - y + 3 = 0 \) is a straight line with slope 2.

What are the points where the line has this slope?

\( y' = 2x - 2x^2 = 2 \Rightarrow x = 2 \Rightarrow y = 0, v = 2 \)

\( y(0) = \frac{4}{3} \)

\( y_1 = \frac{4}{3} 2x + \)

\( y(2) = 4 \)

\( y_2 = 4 + 2(x-2) \)

\( \left( \frac{3}{x^3} \right) - x^2 + 2x + \frac{4}{3} \)

Question 21: Calculate the derivative of the function \( y = x^4 \cos \)

Answer: \( (4x^2 \cos x)' = 8x \cos x - 4x^2 \sin \)

Question 22: Calculate the derivative of the function \( y = x + x^2 \sin 2x \cos x \)

Answer: \( (x^2 \sin x + 2x \cos x)' = x + 2x \sin x^2 \cos x + x - 2x \cos 2x \sin x = x + 2x \cos x^2 \cos x \)

Question 23: Calculate the derivative of the function \( y = \cos x \tan \)

Answer: \( D_\tan x (\cos x \tan x) = - \sin x \tan x - \cos x \cot x \)

Question 24: Calculate the derivative of the function: \( z = (\cot y) \left( \frac{1}{1 - \sin y} \right) \)

Answer: \( (d / dy) \left( \frac{\cot y}{1 - \sin y} \right) = \left( \frac{\csc^2 y (1 - \sin y) + \cot y \cos x}{1 - \sin y} \right) \)

Question 24: Calculate the derivative of the function at the given point: \( f(x) = x \sin x \)

Answer: \( f'(x) = \sin x + x \cos x \)

\( f'\left(\frac{(3\pi)}{2}\right) = \sin \left(\frac{(3\pi)}{2}\right) + (\frac{(3\pi)}{2}) \cos \left(\frac{(3\pi)}{2}\right) = -1 \)
Question 25: Compute the derivative of the function at the point indicated: \( f(x) = \cos x + 1 \) (\( x - 1 \) xsin)

Answer:

\[
f'(x) = -\sin x (x\sin x - 1) + (\cos x + 1) (\sin x + x\cos x)
\]

\[
f'\left(\frac{\pi}{2}\right) = -\sin \left(\frac{\pi}{2}\right) \left(\frac{\pi}{2}\right) \sin \left(\frac{\pi}{2}\right) - 1 + (\cos \left(\frac{\pi}{2}\right) + 1) \left(\sin \left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right) \cos \left(\frac{\pi}{2}\right)\right) 2 = \frac{1}{2} \pi
\]

Question 26: Calculate the equation of the line tangent to the graph of the function, the indicated point: \( f(x) = \cos x \)

Answer:

\[
f'(x) = -\sin x
\]

(A) \( x = \frac{\pi}{2} \)

\[
f'\left(\frac{\pi}{2}\right) = -\sin \left(\frac{\pi}{2}\right) = -1
\]

\[
f\left(\frac{\pi}{2}\right) = \cos \left(\frac{\pi}{2}\right) = 0
\]

\[
y = - (x - \left(\frac{\pi}{2}\right))
\]

(B) \( x = -\frac{\pi}{2} \)

\[
f'\left(-\frac{\pi}{2}\right) = -\sin \left(-\frac{\pi}{2}\right) = 1
\]

\[
f\left(-\frac{\pi}{2}\right) = \cos \left(-\frac{\pi}{2}\right) = 0
\]

\[
y = (x + \left(\frac{\pi}{2}\right))
\]

(C) \( x = \frac{\pi}{6} \)

\[
f'\left(\frac{\pi}{6}\right) = -\sin \left(\frac{\pi}{6}\right) = -\frac{1}{2}
\]

\[
f\left(\frac{\pi}{6}\right) = \cos \left(\frac{\pi}{6}\right) = \frac{1}{2} \sqrt{3}
\]

\[
y = \frac{1}{2} \sqrt{3} - \frac{1}{2} (x - \left(\frac{\pi}{6}\right))
\]

Question 27: Calculate the derivative of the function \( y = (10-5x)^4 \)

Answer: \( \frac{d}{(dx)} (10-5x)^4 = 2500 (x-2)^3 \)

Question 28: Calculate the derivative of the function \( y = \sin x^2 \)

Answer: \( \frac{d}{(dx)} (x^2 \sin) = x^2 2x \cos \)

Question 29: Calculate the derivative of the function \( y = \cos (3x^2 + 1) \)

Answer: \( \frac{d}{(dx)} (\cos (3x^2 + 1)) = -6x \sin (3x^2 + 1) \)

Question 30: Calculate the derivative of the function \( y = (x^2-4x^{-2})^2 (x^2 + 1)^{-1} \)

Answer: \( \frac{d}{(dx)} ((x^2-4x^{-2})^2 (x^2 + 1)^{-1}) = \frac{1}{2} + + 2x^6 8x^4-48x^2-32 \)

Question 31: Calculate the derivative of the function \( y = (((2x^2 + 1) / (3x^3 + 1)))^2 \)

Answer: \( \frac{d}{(dx)} (((2x^2 + 1) / (3x^3 + 1)))^2 = 2 (x / (3x^3 + 1)^3) (12x^5 + 24x^3-8x^2 + 9x-4) \)
Question 32: Calculate the derivative of the function \( y = (x^2 \sin x - 3 \cos x)^3 \)

Answer: \( \left( \frac{d}{dx} \right) (2 \sin x - 3 \cos x)^3 = 3 (x + 2 \cos 3 \sin x) (x - 2 \sin 3 \cos x) \)

\( \left( \frac{d}{dx} \right) (\sin^2 (\cos 2x)) = -4 \sin 2x \cos (\cos 2x) \sin (\cos 2x) \)

Question 33: Calculate the derivative of the function \( y = (5 - 2x^2)^{\frac{1}{3}} \)

Answer: \( \left( \frac{d}{dx} \right) (5 - 2x^2)^{\frac{1}{3}} = \frac{4}{3} \frac{x}{(5 - 2x^2)^{\frac{4}{3}}} \)

Question 34: Calculate the derivative of the function \( y = \sqrt{1 + \csc^2 x} \)

Answer: \( \left( \frac{d}{dx} \right) \sqrt{1 + \csc^2 x} = -\sqrt{2} \frac{\csc x}{(\sin^3 x) \sqrt{-\left( \frac{1}{\sin^2 x} \right) (\cos 2x - 3)}} \)

Question 35: Calculate the derivative of the function \( y = \sqrt[3]{(3x^2 + 5x - 1)^2} \)

Answer: \( \left( \frac{d}{dx} \right) \sqrt[3]{(3x^2 + 5x - 1)^2} = \left( \frac{6x + 5}{(3x^2 + 5x - 1)^2} \right) \)

Question 36: Calculate the derivative of the function \( y = (\sin \sqrt{x} \cos \sqrt{x}) \)

Answer: \( \left( \frac{d}{dx} \right) (\sin \sqrt{x} \cos \sqrt{x}) = \frac{1}{3x} \cos 2 \sqrt{x} \sqrt{x} \)

Question 37: Calculate the implicit derivative of the equation: \( 4x^2 - 9y^2 = 1 \)

Answer: \( 8x - 18 yy' = 0 \Rightarrow y' = \left( \frac{4x}{9y} \right) \)

Question 38: Calculate the implicit derivative of the equation: \( 2x^3 y^3 + 3xy^3 + 5 = 0 \)

Answer: \( 2x^3 y' 3y^3 + + + 9 6x^2 y xy^2 y' = 0 \Rightarrow y' = -\left( \frac{6x^2 y 3y^3 +}{+ 2x^3 9xy^2} \right) \)

Question 39: Calculate the implicit derivative of the equation: \( y + \sqrt{xy} = 0 - 3x^3 \)

Answer: \( y' + (1 / (2 \sqrt{xy})) y + (1 / (2 \sqrt{xy})) xy' - 9x^2 \Rightarrow y' = \left( \frac{18x^3 \sqrt{xy}}{\sqrt{(xy) - 2y}} / (x^2 - 4 xy) \right) \)

Question 40: Calculate the implicit derivative of the equation: \( y \sqrt{2 + 3x} + x \sqrt{1 + y} - x = 0 \)

Answer: \( y' (\cot^2 (xy) + 1) y' (\cot^2 (xy) + 1) xy' + xy' = 0 \Rightarrow y' = \left( \frac{(y' (\cot^2 (xy) + 1) y)}{x (\cot^2 (xy)))} \right) \)

Question 41: Calculate the implicit derivative of the equation: \( y / (2 + 3x) + x / (1 + y) - x = 0 \)

Answer: \( (3/2) (y / (\sqrt{(3x + 2)})) + \sqrt{(y + 1) + y' \sqrt{(3x + 2) + (1/2) (x / (\sqrt{(y + 1)}))} y' - 1 = 0 \)

\( y = \left( \frac{(1 - (3/2) (y / (\sqrt{(3x + 2)}))) + \sqrt{(y + 1) / (\sqrt{(3x + 2)} + (1/2) (x / (\sqrt{(y + 1)})))} y' - 1}{0} \)
Question 43: Calculate the equation of the line tangent to the normal curve: $9x^3 - y^3 = 1$ at the point $(1,2)$

Answer:

$27x^2 - 3y^2 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{27x^2}{3y^2}$

$y'(1) = \frac{27(1)^2}{3(2)^2} = \frac{9}{4}$

$y = 2 + \frac{9}{4} (x-1)$

Question 44: Calculate the polynomial of nth degree Taylor with the rest of Lagrange: $f(x) = e^{-x}; x_0 = 0; n = 4$

Answer:

The Taylor polynomial of nth degree with the rest Lagrangian is given by:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{(f^{(n)}(x_0))}{(n!)}(x-x_0)^n + \frac{(f^{(n+1)}(\xi))}{((n+1)!)}(x-x_0)^{n+1}$$

where $\xi$ esté between $x_0$ and $x$.

and $\{x\} \times f(0) = 1$

$((D / (dx)) (e^{x}) (x - x_0)) = - e^{x} \times f'(0) = -1$

$((D^2) / (dx^2)) (e^{x}) (x - x_0)) = e^{x} \times f'(0) = 1$

$((D^3) / (dx^3)) (e^{x}) (x - x_0)) = - e^{x} \times f''(0) = -1$

$((D^4) / (dx^4)) (e^{x}) (x - x_0)) = e^{x} \times f'''(0) = 1$

$((D^5) / (dx^5)) (e^{x}) (x - x_0)) = - e^{x} \times f^{iv}(0) = -1$

$f(x) = x^{1} + (1/2) x^{2} - (1/6) x^{3} + (1 / (24)) x^{4} + ((E^{x}) \xi) / (120) x^{5}$

Question 45: Prove the following statements ::

Answer: $(c) = 0$ where $c$ is a constant

$\lim_{h \to 0} ((cc) / h) = h \lim_{h \to 0} (0 / h) = 0$

Question 46: Prove the following statements:

Answer: $(x^n)' = nx^{n-1}$ to $n \in \mathbb{Z}$

$\lim_{h \to 0} ((x + h)^n - x^n) / h \lim_{h \to 0} (x^n nx^{n-1}h + h^2 + (F(x, h)) - x^n) / h$

$\lim_{h \to 0} (0) [nx^{n-1} + h (F(x, h))] = nx^{n-1}$

Question 47: Prove the following statements ::

Answer: $(u^r)' = r u^{r-1}$ for $r \in \mathbb{Q} \setminus (x) = 0$

Generalization 3 for fractional exponents and chain rule
Unity 3

Question 1: Find the function of extremes: \( f(x) = \sqrt{4-x^2} \)

Answer:

\[ f(x) = \sqrt{4-x^2} \text{ in } [-2,2] \]

\[ f'(x) = \frac{4x^2}{2\sqrt{4-x^4}} \], then \( f \) is differentiable \([-2,2]\)

\[ f'(x) = 0 \Rightarrow 4x^3 = 0 \Rightarrow x = 0 \]

\[ \lim_{x \to -2^+} \sqrt{4-x^2} = 0 \]

\[ \lim_{x \to 2^-} \sqrt{4-x^2} = 0 \]

\[ f(0) = 2 \]

The absolute minimum does not exist and the absolute maximum of the function is 2 obtained at \( x = 0 \)

Question 2: Prove Rolle’s theorem to the function \( f(x) = x^3 + x^2 - x \) in the range \([-2,1]\)

Answer:

\( f \) is continuous in \([-2,1]\) (polynomial)

\( f \) is differentiable because \( f'(x) = 2x-1 + 3x^2 \) is continuous \([-2,1]\)

Then there \( c \in [-2,1] \) such that \( f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - (-2)}{3} = 1 \)

\[ 3c^2 + 2c - 2c + 1 = 1 \Rightarrow 3c^2 = 0 \Rightarrow c = \pm \frac{1}{3}\sqrt{7} \]

Both are in the interval \([-2,1]\).

Question 3: Ensure the mean value theorem to the function \( f(x) = \frac{2x-1}{3x-4} \) in the range \([1,6]\)

Answer:

\( \text{domf } R = \{4/3\} \)

\( f \) is not continuous in the range given \( \{2x-1\}/(3x-4) \)

Question 4: Find the extreme and sketch graph \( f(x) = x^{(2/3)}(x-1)^2 \)

Answer:

\[ f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(1-x)^2 + 2x^{\frac{2}{3}}(x-1) = 0 \]

\[ (2/3)x^{-\frac{1}{3}}(1-x)^2 + 2x^{\frac{2}{3}}(x-1) = 0 \Rightarrow (2/3)(x-1)^2 + 2x(x-1) = 0 \]

\[ (8/3)x^2 - (10/3)x + (2/3) = 0 \Rightarrow 4x^2 + 5x - 1 = 0 \]

\( x \left(1/4\right) \)
(X -1) - - - + 0
(X - (1/4)) - 0 + + +
(X - 1) (x - (1/4)) + 0 - 0 +

In (1/4), the sign is positive before and after the negative, then a local maximum is 1, the signal is negative before and positive after, so it is a local minimum

x ^ \{ (2/3) \} (x -1)^2

Question 5: Find the extreme and sketch the graph of the function f(x) = \tan x

Answer:

f'(x) = \frac{d}{dx}(\tan x) = 1 + \tan^2 x \sec^2 x

f''(x) = \frac{d}{dx}(\tan^2 x + 1) = 2(\tan x)(\tan^2 x + 1) = 0

\tan x = 0 \Rightarrow x = 0, \pm \pi, ...

The tangent slope when \tan x = 0 is f'(0, \pm \pi, ...) = 1
Unit 3. Integral Calculus in R

Introduction to the Unit, Introduction should be narrative

This unit provides the following:

- Primitives of elementary functions
- The integral of a differential form
- The fundamental theorem of calculus
- Integration techniques
- The Riemann integral
- The geometric interpretation of the Riemann integral

Areas of calculus and elementary volumes using integration. These are indispensable tools in the application of the calculation method, leading the study here. If present, also some applications, including the calculation of areas of flat figures, applying the calculation method.

Unit Objectives

After completing this unit, you should be able to:

- identify a function primitives,
- identify differential forms,
- calculate integrals and
- solve ordinary differential equations in 1st order
- apply areas of plane figures bounded by function graph and relate this amount with definite integral through the Fundamental Theorem of Calculus

Key Terms

**Primitivation terms**: Reverse operation of derivation (except constant)

**Primitive f**: If a derivative is equal f

**Exterior derivative**: $dF = F'(x) \, dx$

**prop.Leibniz**: $d(FG) = + GdF \, FdG$

**deriv. ext. F ∘ h**: $d(F ∘ h) = F'h'dx = F'dh$

**prop.Poincare**: $d(dF) = 0$
**Differential form:** $g(x) \, dx$ where $g(x)$ is defined in an open related

**Integration:** inverse operation of the exterior derivative (excp const.)

**Integ.immediate:** recognize in the derivation

**Integtable.parts**, using prop. Leibniz

**Integ.Replace:** using prop. composite function

**Integral curve:** differential equation

**sum solution:** $\sum$ symbol representing sum indexed

**partial sums installments:** $s_\{n\} = x_1 \, x_2 \, \cdots \, + \, x_{\{n\}} = \sum_{\{k \, = \, 1\}}^{\{n\}} x_{\{k\}}$

**infinite series:** $\sum_{\{k \, = \, 1\}}^{\{\infty\}} x_{\{k\}}$ sum of endless plots

**difference(rear):** $\Delta s_{\{n\}} = s_{\{n\}} - s_{\{n-1\}}$ to $n \in \mathbb{N}$

**partition (mesh):** set of points, including the interval

**regular grid lips:** equidistant points

**step mesh:** distance between points on a regular mesh $h = \frac{ba}{n}$

**greater sums of Riemann:** $\sum_{\{i \, = \, 1\}}^{\{n\}} M_{\{i\}} \, \Delta_{\{i\}} \, x$

**lower sums of Riemann:** $\sum_{\{i \, = \, 1\}}^{\{n\}} m_{\{i\}} \, \Delta_{\{i\}} \, x$

**fnintegrable:** (Riemann) limits of Riemann sums are

**regstripes.horizontally simple:** can subdivide in horizontal bands
Learning Activities

Activity 1 - Primitives and integration in R

Introduction

This activity provides the following:

- Primitives (antidifferentiation) functions.
- Derivative outside.
- Differential forms.
- Integration of differential forms.
- Integration techniques.
- Ordinary Equations in 1st order differences.

Activity Details

Early functions in R

Other than a constant, the primitivation is the inverse operation of derivation (formal definition is given below).

It is said that \( G(x) \) is a primitive \( g(x) \) \( g'(x) = g(x) \) and \( g(x) \) is a primitive \( g(x) \) then \( g(x) + c \) is also a primitive \( g(x) \) for any constant \( c \).

In demonstration simply use the fact that the derivative of a constant function is zero

\[
(g(x) + c)' = g'(x) + (c)
\]

\[= g'(x) \]

\[= g(x) \]

By definition, if \( f'(x) \) is a derivative of \( f(x) \) then \( f(x) \) is a primitive \( f'(x) \).

Immediate primitives are obtained from the derived table, making reading from right to left and adding a constant \( c \)

Example 1: \( g(x) = x^3 + c \) is a primitive \( g(x) = 3x^2 \) as

\[g'(x) = (x^3 + c) +\]

\[= 3x^2 + (c)'
\]

\[= g(x) \]
Example 2: \( L(x) = \sqrt{x} + c \) is a primitive \( g(x) = \frac{1}{2\sqrt{x}} \) as 
\[
g'(x) = (\sqrt{x} + c)' = (x^{(1/2)})' + (c)' = \frac{1}{2\sqrt{x}} = g(x)
\]
However, because the domain of \( f'(x) \) is a subset of \( f(x) \)
\[\text{dom}(f') \subseteq \text{dom}(f)\]
requires that the field of primitive \( g(x) \) be a superset of the domain \( g(x) \)
\[\text{dom}(g) \supseteq \text{dom}(g)\]
the primitive of a function \( g(x) \), represented by \( g(x) \) is a class of functions that are distant one another of a constant such that \( g'(x) = g(x) \) and \( g \) domain \( (x) \) is a maximal superset of the \( g \) domain \( (x) \).

Example 1: primitive \( g(x) = \frac{1}{x} \) \( G(x) = \ln|x| + c \) as 
\[
G'(x) = (\ln|x| + c)' = (\ln|x|)' + (c)' = \frac{1}{x} = g(x)
\]
and the domain \((\ln|x| + c)\). is \( \mathbb{R}/\{0\} \), equal to the domain \((1/x)\).

The primitivation functions in \( \mathbb{R} \) enjoy the following properties: if \( F(x) \) and \( G(x) \) are \( f \) primitives \( f(x) \) and \( g(x) \) respectively, then:

- Primitive at any constant is zero because \( (c) = 0 \) for any constant \( c \),
- \( F'(x) = f(x) \) by setting,
- \((d / (dx)) F(x) = f(x) \) (previous expression Leibniz notation)
- \((\lambda F(x) + \beta G(x)) \) is the primitive \( \lambda f(x) + \beta g(x) \) since the linearity of the junction
- \((\lambda F(x) + \beta G(x)) = \lambda f'(x) + \beta G'(x) = \lambda f(x) + \beta g(x) \)
- \( F(x) G(x) \) is a primitive \( F'(x) G(x) + F(x) G'(x) \) because the Leibniz rule: \( (F(x) G(x))' = F'(x) G(x) + F(x) G'(x) \)

**Derived Exterior and Differentially**

Given a function \( f(x) \) differentiable an open interval \( I \subseteq \mathbb{R} \) the exterior derivative \( F(x) \) is defined as
\[
dF = F'(x) \, dx.
\]
For \( f(x) G(x) \) and continuous functions \( f(h(x)) \) compositave is an open \( I \subseteq \mathbb{R} \) range, and scalar \( \alpha, \beta \in \mathbb{R} \) the exterior derivative satisfies the following properties:
The statement verifies that the first three properties derive from the exterior derivative of definition and its properties to those derived are as follows:

\[
\begin{align*}
\text{d} (\alpha F + \beta G) &= (\alpha F + \beta G) \, dx \\
&= \alpha F' \, dx + \beta G' \, dx \\
&= \alpha \text{d}F \beta \text{d}G + \\
\text{d} (FG) &= (FG) \, dx \\
&= (GF' + FG') \, dx \\
&= GF' \, dx + FG' \, dx \\
&= + G \text{d}F \, F \text{d}G \\
\text{d} (F \circ h) &= (F \circ h)' \, dx \\
&= F'h' \, dx \\
&= F' \, dh
\end{align*}
\]

The last property is derived from the two other properties

\[
\begin{align*}
\text{d} (\text{d}F) &= \text{d} (F' \, dx) \\
&= dF' \, dx + F'' \, dx \\
&= + F'' \, dx \, F' \, dx \\
&= 0
\end{align*}
\]

is designated as differential form (grade 1) in an open \( I \subset \mathbb{R} \) the expression

\[ g(x) \, dx \]

where \( g(x) \) is a continuous function of \( I \).

NB://

A differential form \( g(x)dx \) and an open \( I \subset \mathbb{R} \) is said to be accurate if it can be written as an outer derived from some function \( g(x) \) \( g(x) \, dx = dG \)

A differentially \( \sigma \) is said to be closed if its derivative is zero outside \( d\sigma = 0 \)
In an open connection \( I \subset \mathbb{R} \), a differential form \( g(x)dx \) is exact if it is closed.

In the statement, consider the differential form \( g(x)dx \) an open connection \( I \subset \mathbb{R} \).

If \( g(x)dx \) is accurate then there is some function \( g \) in \( I \subset \mathbb{R} \) such that \( g(x)dx \ dG \), then:

\[
d(g(x)dx) = d(dG) = 0
\]

If \( g(x)dx \ d \) is then closed \( (g(x)dx) = 0 \), \( \log o g(x)dx = cdx \), for some constant \( c \). Let \( G = cx \) then:

\[
dG = cdx
\]

soon \( g(x)dx \) is correct.

By definition, if \( g(x)dx \ dG \) for any \( L(x) \) function on the interval \( I \), then \( G'(x) = g(x) \)

**Integral of differential forms**

Other than a constant, integration (indefinite) is the inverse operation of foreign derivation:

\[
\int dF = F + c
\]

it follows easily the relationship between primitive functions and integrals of differential forms

If \( F(x) \) is the primitive of \( f(x) \) then

\[
\int f(x) \ dx = F(x) + c
\]

open and connected num interval \( \mathbb{R} \).

The statement is made as follows: if \( F(x) \) is the primitive \( F(x) \) then:

\[
dF = F'(x) \ dx
\]

\[
= f(x) \ dx
\]

Once

\[
F(x) = \int dF(x) = \int f(x) \ dx
\]

The integrating differential forms enjoys the following properties deducted directly from the functions of primitivation properties:

\[
(\int f(x) \ dx)' = f(x) \text{ the definition}
\]

\[
(d / (dx)) \int f(x) \ dx = f(x) \text{ (with Leibniz notation)}
\]

\[
\int 0 \ dx = c \text{ because } (c) = 0 \text{ for any constant } c.
\]

\[
\int \lambda f(x) \ dx = \lambda \int f(x) \ dx + c \text{ because the linearity of derivation}
\]

\[
(\lambda \int f(x) \ dx + c)' = \lambda (\int f(x) \ dx)' + (c) + = \lambda f(x)
\]

\[
\int (\beta f(x) + \beta g(x)) \ dx = \alpha \int f(x) \ dx + \beta \int g(x) \ dx + C \text{ for linearity of derivation}
\]
By definition, primitives are differentiable, so primitive functions are continuous in $I \subset \mathbb{R}$ range.

If a function is discontinuous at some point within an interval $I$, then it is not primitive any function in that range.

In the demo let an arbitrary function $G(x)$ is discontinuous at $a \in \text{int}(I)$

$$\lim_{x \to a^-} G(x) \neq \lim_{x \to a^+} G(x)$$

Suppose that $G(x)$ is the primitive $g(x)$. By definition, $g'(x) = g(x)$, for $\forall x \in I$, then $g'(a) = g(a)$, which contradicts the assumption.

**Integration Techniques**

Calculation of primitive boils down to transform a differential form given, $f(x)dx$, an exact differential form, $dF$, and primitive by using the definition

$$\int dF(x) = F(x) + C$$

To find an exact differential form we use integration techniques, among which are:

- Immediate integration
- Integration for parts
- Integration by substitution

**Immediate Integrals**

Some exact differential forms can be recognized by reading (unlike) the table of outstanding derivatives:

$$0 DX = AD$$

$$= dx 1 DX$$

$$x^a dx d = (x^{a+1}) / (n + 1))$$ for $n \in \mathbb{N} \ ex > 0$ is the irreducible form the denominator of $q \in \mathbb{Q}$ for par.

$$\int (1 / (2|x|)) dx = d\sqrt{x} \ for \ x > 0$$

$$\cos x D x = x DSIN$$

$$-\sin x D x = x DCOS$$

$$\sec^2 x dx \ dtan = x \ in \ its \ domain$$

$$\tan x \cdot \sec x D x = DSEC x$$

$$\int (1 / (1 - x^2)) dx = \arcsin x$$

$$\int (1 / (1 - 1^2)) dx = \arccos x$$

$$\int (1 / (1 + x^2)) dx = \arctan x$$

$$e^x dx = a^x (\text{remarkable property})$$

$$a^x dx = d ((a^x) / (\ln a)) \ for \ a > 0$$
\[ \int \frac{1}{x} \, dx = \ln |x| \quad \text{for } x \neq 0 \]

The integration of differential medium is immediately calculated using the definition:

**Example 1:** Calculate the integral of differentially \( x \, dx \)

Recognizing (immediately) that \( x \, dx = \frac{(x^2)}{2} \) is accurately and applying the definition

\[
\int x \, dx = \int \frac{(x^2)}{2} = \frac{(x^2)}{2} + C
\]

**Example 2:** Calculate the integral of the differential form \( \frac{1}{\sqrt{x}} \)

Recognizing (almost immediately) that \( \frac{1}{\sqrt{x}} \, dx = d(\sqrt{x}) \) for \( x > 0 \) is an exact form and applying the definition

\[
\int \frac{1}{\sqrt{x}} \, dx = \int d(\sqrt{x}) = C + 2\sqrt{x}
\]

**Integration by Parts**

An exterior derivative of a product of two functions \( F(x) \) and \( G(x) \), differentiable in \( \mathbb{R} \) an open interval: \( d(FG) = GdF + FdG \) can be very useful to calculate accurate differential forms.

There are two parts per integration methods:

**Part I:** If formed, \( f(x) \, dx \), is identical to one of the parties \( G \cdot dF \) while the other part is an accurate way \( F \cdot dG = d(H) \), integrate up the difference of the two exact forms

\[
G \cdot dF = d(FG) - F \cdot dG
\]

**Part II:** If the given form \( f(x) \, dx \), there is a “part” which recognizes a form accurate, identifies this “party” with \( dF \) and the remainder with \( G \) method of application examples in Part I.

**Example 1:** Calculate the integral of the function \( f(x) = \ln x \)

The plan is transform \( \ln x \, dx \) as an exact differential form and integrate it using the definition

Recognizing the similarity of \( \ln x \, dx \) with \( G \cdot dF \), used \( FG = x \ln x \) for

\[
d(x \ln x) = \ln x \, dx + x \, d\ln x
\]

\[
= \ln x \, dx + x \left(\frac{dx}{x}\right)
\]

\[
= \ln x \, dx + dx
\]

then the differential form given is the difference of two exact forms and the linearity of the exterior derivative

\[
\ln x \, dx = d(x \ln x) - dx
\]

\[
= d(x \ln x) - dx
\]
it becomes clear then that
\[
\int \ln x \, dx = \int d \left( x \ln x \right)
\]
\[
= x - x \ln + C
\]
and it appears infact that
\[
(x - x \ln + C) \,' = \ln x + 1 - 1 + 0 = \ln x
\]

Example 2: Calculate the integral of the function \( f(x) = x^x \), for \( a > 0 \).
the plan is transform \( x^x \, dx \) as an exact differential form.
\[
d \left( x^x \right) = a^x \, dx + x \, da^x
\]
\[
= a^x \, dx + x^x \ln a \, dx
\]
As \( ^x \, dx = (1 / (\ln a)) \) of \( ^x \)
\[
d \left( x^x \right) = d \left( (a^x / (\ln a)) + \ln a \cdot x \right) \, dx
\]
obtains the difference of two forms and the exact linearity of the exterior derivative
\[
x^x \, dx = (1 / (\ln a)) \, d \left( x^x - (a^x / (\ln a)) \right)
\]
it becomes clear then that
\[
\int x^x \, dx = (1 / (\ln a)) \int d \left( x^x - (a^x / (\ln a)) \right)
\]
\[
= (1 / (\ln a)) \left( x^x - (a^x / (\ln a)) \right) + C
\]
and there in fact that
\[
\left( (a^x / (\ln a)) - (1 / (\ln a)) \right) \cdot \left( x^x - (a^x / (\ln a)) \right) + (a^x / (\ln a)) + 0 = x^x
\]

Example 3: Calculate the integral of the function \( f(x) = \sec^2 x \) in the interval \( \left[ -\pi/2, +\pi/2 \right] \)
\[
d(tan x) = (x - \sin \cos x \, dx \sin x \, d\cos x) / (\cos^2 x) = dx / (\cos^2 x) = \sec^2 x \, dx
\]
Therefore
\[
\int \sec^2 x \, dx = d = (\tan x)
\]
\[
= x \tan C +
\]

Example 4: Calculate the integral of the function \( f(x) = x \cos \):
\[
d(x \sin x) = \sin x \, dx + x \, d\sin x
\]
\[
= d (-\cos x) + x \, d \, x \cos
\]
Logo
\[
\int x \cos x \, dx = \int d \left( x \sin x + \cos x \right)
\]
\[
= \cos x + x + C \sin
\]
And it turns out the fact that
\[
(\cos x + x \sin x + C) \,' = -\sin x + \sin x + x \cos x + 0 = x \cos;
\]
Example 5: Calculate the integral of the function $f(x) = x \sin x$:

$$
d (xcos x) = \cos x \, dx + x \, d\cos x
= d (\sin x) \, x \, d\sin x
$$

Logo

$$\int x \sin x \, dx = \int d (\sin x - x \cos x)
= \sin x - x \cos x + C$$

And there is the fact that

$$(\sin x - x \cos x + C)' = \cos x \cos x + x \sin x + 0 = x \sin x$$

Method of application examples (II)

Example 1: Calculate the integral function $f(x) = x^x$
to make $x^x \, dx$ an exact differential form, choose a “part” which recognizes an exact form, eg $e^x \, dx = a^x$, identified with $dF = e^x \, dx = a^x$, then $F = e^x$ and the other party $x = G$

gets up so

$$d (x^x) = x \, d(e^x) + e^x \, dx
= x^x \, dx + of^x$$

gets the difference of two exact forms and the linearity of the exterior derivative

$$x^x \, dx = d (x^x - e^x)$$

it is clear then that

$$\int x^x \, dx = \int d (x^x - e^x)
= x^x - e^x + C$$

Example 2: Calculate the integral of the function $f(x) = x^2 e^x$

As before, identifies $dF = e^x \, dx = a^x$, then $F = e^x$ and $G = x^2$

gets up so

$$d (x^2 e^x) = x^2 \, de^x + e^x \, dx^2
= x^2 e^x \, dx + 2xe^x \, dx$$

The previous result

$$2xe^x \, dx = d (2xe^x \, \{x\}^2 - 2xe^x + 2e^x)$$

gets the difference of two exact forms and the linearity of the exterior derivative

$$x^2 e^x \, dx = d (x^2 e^x \, \{x\}^2 - 2xe + 2e^x)$$
it is clear then that

\[ \int x^2 e^x \, dx = \int d(x^2 e^x - 2xe^x + 2e^x) \]
\[ = x^2 e^x - 2xe^x + 2e^x + C \]

And it turns out in fact that

\[ (x^2 e^x - 2xe^x + 2e^x + C)' = 2xe^x + x^2 e^x - 2e^x - 2xe^x + 2e^x + 0 = x^2 e^x \]

Integration by Substitution (Replacing)

The exterior derivative of a composition \( F \circ g \) functions where \( F(g) \) is differentiable in their field \( g(x) \) is a differentiable bijection an open interval \( I \subset \mathbb{R} \)

\[ d(F \circ h) = F'(h) \, dh = F'(h') \, dx \]

can be very useful to calculate exact differential forms.

Given a form \( f(x) \, dx \), aiming for a "pullback" of the domain of \( f \), through a post-makeup \( u = h(x) \) or pre-composition \( x = g(u) \), designated respectively forward and backward substitution:

Substitution

\[ u = h(x) \quad x = g(u) \]
\[ du = h'(x) \, dx \quad dx = g'(u) \, du \]

In direct displacement, it becomes \( u = h(x) \) and \( du = h'(x) \, dx \) becomes the form given in \( (f(x) \, h'(x)) \, dx \) and using the substitution \( x = h^{-1}(u) \) in the expression \( (f(x) / h'(x)) \) to achieve a form \( K(u) \, du \) to be possibly recognized as an exact differential manner.

Examples:

Calculate the integral of the function \( f(x) = (1 / (1 + e^x)) \)

Let

\[ u = e^x \]

Then

\[ du = e^x \, dx \]

As \( u = e^x \) is a diffeomorphism has inverse differentiable

\[ x = \ln u \quad \text{for } u > 0 \]

Using the inverse in the expression \( (f(x)) / (h'(x)) = (1 / ((1 + e^x) e^x)) \) is obtained

\[ (1 / ((1 + u) u)) = (1 / u) - (1 / (1 + u)) \] whose shape is recognizable

\[ ((1 / u) - (1 / (1 + u))) \, du = d (\ln u - \ln (1 + u)) \]
\[ = d (\ln u - \ln (1 + u)) \quad \text{for } u > 0 \]
Integrating and replacing back the variable $x$, obtain

$$\int \frac{1}{1 + e^x} \, dx = \int \frac{1}{u} \left(1 - \frac{1}{1 + u}\right) \, du$$

$$= u \ln(1 + u) + C$$

$$= x - \ln(1 + e^x) + C$$

In reverse substitution, it is $x = g(u)$ and $dx = g'(u) \, du$ turns the form given $f(x) \, g'(u) \, du$ and replacement it uses $x = g(u)$ on the expression $f(x) \, g'(u)$ to achieve a form $K(u) \, du$ to be possibly recognized as an accurate differentially.

Examples - Indirect substitution

Calculate the same integral of the function $f(x) = \frac{1}{1 + e^x}$ (by the method II)

Let

$$x = \ln u \text{ to } u > 0$$

then

$$dx = \frac{1}{u} \, du$$

Using the substitution expression $f(x) \, g'(u) = \frac{1}{1 + e^x} \frac{1}{u}$ is given by $\frac{1}{u - (1 + u)} = (1/u) - (1/(1 + u))$ whose shape is recognizable

$$= \frac{1}{u - (1 + u)} \, du = d(\ln |u| - \ln |u + 1|)$$

$$d = (U - \ln(1 + u)) \text{ as } u > 0$$

Integrating and substituting back the variable $x$, obtain

$$\int \frac{1}{1 + e^x} \, dx = \int \frac{1}{u} \left(1 - \frac{1}{1 + u}\right) \, du$$

$$= \int d \left(\ln u - \ln(1 + u)\right)$$

$$= \ln u - \ln(1 + u) + C$$

$$= x - \ln(1 + e^x) + C$$

Equations linear ordinary differential 1st order

Linear differential equation of 1st order is a relationship between the different forms of two variables $x, t$ set in an open

$$(x, t, dx, dt) = 0$$

explicitly setting of the variables a function other $x(t)$, obtain $(dx)/(dt) = f(x, t)$

whose solution is an integral curve $x(t)$ defined in an interval $I$.

In the special case of the function $f(x, t)$ has representation

$$(dx)/(dt) + p(t) \cdot x = q(t)$$
the equation is said to be linear

(R) For the case \( p(t) = 0 \)
the differential equation reduces to \( (dx) / (dt) = q \)
the term \( dx = QDT \) is an exact differential form and can be integrated
\[
\int q dt \int dx = x + C
\]
the integral curves of linear differential equation of 1st order depend on the parameter \( C \)
\[
x = C \int q dt
\]
Example: Calculate the integral curves of the differential equation \( (dx) / (dt) = t^3 t^2 \)
For previous examples, leaving
\[
u = 3t \Rightarrow du = 3dt
\]
obtained:
\[
te^{3t} dt = (u / 9) \text{ and } \int u du
\]
\[
= (1/9) u \int u dt
\]
\[
= (1/9) d (u^u - e^u)
\]
\[
x(t) = \int te^{3t} dt
\]
\[
= (1/9) \int d (u^u - e^u)
\]
\[
= ((e^u) / 9) (u-1) + C
\]
\[
= ((e^{3t}) / 9) (3t-1) + C
\]
(II) In the case \( p(t) \neq 0 \)
search is complete, in a detailed form the differential equation
\[
(dx) / (dt) + px = q
\]
Making \( p = (\mu \prime / \mu) \) for \( \mu(t) \neq 0 \) then it would have been the derivative of a product
\[
(dx) / (dt) + (\mu \prime / \mu) = q x
\]
\[
\mu (dx) / (dt) + = \mu \prime x \mu q
\]
\[
(\mu x)^{\prime} \mu q =
\]
With the exact differential form
\[
d (\mu x) = \mu q dt
\]
\[
\int d (\mu x) = \int \mu q dt
\]
\[
\mu x = \int \mu q dt
\]
The integral curves are given by

\[ x = \mu^{-1} \int \mu q \, dt + C \mu^{-1} \]

\( \mu \) The function is called by an integral factor and can be explicitly calculated from \( p = (\mu' / \mu) \) with \( \mu(t) \neq 0 \)

\[
(\mu' / \mu) \, dt = pdt
\]

\[
\int d (\ln \mu) = \int pdt
\]

Results in

\[ \ln \mu = \int pdt \]

\[ \mu = e^{\int pdt} \]

Example: Calculate the integral curves of linear EDO 1st order given by

\[ x' - (1 / t) x = -t \]

Identifying

\[ p = - (1 / t) \]

\[ q = -t \]

gets up

\[ \mu = e^{\int - (1 / t) \, dt} \]

\[ = e^{- \ln |t|} \]

\[ = (1 / |t|) \text{ for } t \neq 0 \]

Logo

\[ x = \mu^{-1} \int \mu q \, dt + C \mu^{-1} + \]

\[ = - |t| \int (t / |t|) \, dt + C |t| \]

For \( t > 0 \)

\[ x = -t \int dt + Ct \]

\[ = -t^2 + Ct \]

\[ = t (Ct) \]

To \( t < 0 \)

\[ x = t \int (-1) \, dt - Ct \]

\[ = -t^2 - Ct \]

\[ = -t (C + t) \]
Conclusion

This unit covered the definitions and properties of the primitives operations (antidifferentiation) functions, the exterior derivative of differential and integration of differential forms forms. Also presented integrating differential forms and techniques related to this matter with a resolution of ordinary differential equations 1st order.

Rating

Group at least 5 and a maximum of 10 of these questions in each 1 hour evaluation test without consultations.

Choose different type questions in the preparation of each test by changing the parameters and database to avoid collages.

Each question is worth between 10% and 20% depending on the number of selected questions.

Question 1: Do the following primitives: ∫ (2 / ([3] √ (x))) dx

Answer:

∫ (2 / ([3] √ (x))) dx = 2∫x ^ {- (1/3)} dx
= 2 (((x ^ {(2/3)}) / ((2 / 3)))) + c
= 3 [3] √ (x²) + c

Verification: (d / (dx)) (3 [3] √ (x²) + c) = (2 / ([3] √ (x)))

Question 2: Do the following primitives: ∫ x (x + 1) dx

Answer:

∫ x (x + 1) dx = ∫ (x ^ {((3/2))} + {x ^ (1/2)}) dx
= ((x ^ {(5/2)}) / ( (5/2))) + ((x ^ {(3/2)}) / ((3/2)))c +
= (2/5) x^(5/2) + (2/3) c + x

Verification: (d / (dx)) ((2/5) x^(5/2) + (2/3) x + c) = x (x + 1)
Question 3: Do the following primitives: \( \int \frac{1}{2} t \cos 4t^2 \, dt \)

Answer: \( \int \frac{1}{2} t \cos 4t^2 \, dt \)

Let \( u = 8t \), \( du = 8 \, dt \)

\[
\int \frac{1}{2} t \cos 4t^2 \, dt = \frac{1}{16} \int \cos u \, du
\]

\[
= \frac{1}{16} \sin u + c
\]

\[
= \frac{1}{16} \sin 4t^2 + c
\]

Verification: \( \frac{d}{dt} \left( \frac{1}{16} \sin 4t^2 + c \right) = \frac{1}{2} t \cos 4t^2 \)

Question 4: Do the following primitives: \( \int [3] \sqrt{6-2x} \, dx \)

Answer: \( \int [3] \sqrt{6-2x} \, dx \)

Let \( u = 6-2x \), \( du = -2 \, dx \)

\[
\int [3] \sqrt{6-2x} \, dx = - \frac{1}{2} \int [3] \sqrt{u} \, (-2 \, dx)
\]

\[
= - \frac{1}{2} \int [3] \sqrt{u} \, du
\]

\[
= - \frac{1}{2} \left( \frac{u^{1/3}}{\frac{1}{3}} \right) + c
\]

\[
= - \frac{3}{8} [3] \sqrt{6-2x} + c
\]

Verification: \( \frac{d}{dx} \left( - \frac{3}{8} [3] \sqrt{6-2x} + c \right) = [3] \sqrt{6-2x} \)

Question 5: Do the following primitives: \( \int \sec^2 5x \, dx \)

Answer:

\[
\int \sec^2 5x \, dx = \frac{1}{5} \int \sec^2 5x \, (5 \, dx)
\]

\[
= \frac{1}{5} \int \sec^2 u \, du
\]

\[
= \frac{1}{5} \tan u + c = \frac{1}{5} + c \tan 5x
\]

Verification: \( \frac{d}{dx} \left( \frac{1}{5} + c \tan 5x \right) = \sec^2 5x \)

Question 6: Do the following primitives: \( \int \frac{(x (3x^2 + 1))}{(3x^4 + 2x^2 + 1)^2} \, dx \)

Answer: \( \int \frac{(x (3x^2 + 1))}{(3x^4 + 2x^2 + 1)^2} \, dx \)

Let \( u = +3x^4 + 2x^2 + 1 \Rightarrow du = 4(3x^3 + x) \, dx \)

\[
\int \frac{(x (3x^2 + 1))}{(3x^4 + 2x^2 + 1)^2} \, dx = \frac{1}{4} \int ((4(3x^3 + x) \, dx) / ((3x^4 + 2x^2 + 1)^2))
\]

\[
= \frac{1}{4} \int (du) / (u^2)
\]

\[
= \frac{1}{4} (\frac{u^{-1}}{-1}) + c
\]

\[
= - (1 / (3x^4 + 2x^2 + 1))) + c
\]
verification: \( \frac{d}{dx} \left( 1 / (3x^4 + 2x^2 + 1) \right) + c \) = \( (3x^3 + x) / ((3x^4 + 2x^2 +1)^3) \)

Question 7: Do the following primitivações: \( \int \left( t + \frac{1}{t} \right)^\frac{3}{2} \left( \frac{t^2-1}{t^2} \right) dt \)

Answer: \( \int \left( t + \frac{1}{t} \right)^\frac{3}{2} \left( \frac{t^2-1}{t^2} \right) dt = \int \left( t + \frac{1}{t} \right)^\frac{3}{2} (1- \frac{1}{t^2}) dt \)

Let \( u = t + \frac{1}{t} \) \( \Rightarrow \) \( du = \left(1- \frac{1}{t^2}\right) dt \)

\[ \int \left( t + \frac{1}{t} \right)^\frac{3}{2} (1- \frac{1}{t^2}) dt = \int u^\frac{3}{2} du \]

\[ = \left( \frac{u^{\frac{5}{2}}}{\frac{5}{2}} \right) + c \]

\[ = \frac{2}{5} \left( t + \frac{1}{t} \right)^{\frac{5}{2}} + c \]

verification: \( \frac{d}{dt} \left( \frac{2}{5} \left( t + \frac{1}{t} \right)^{\frac{5}{2}} + c \right) = \left( \frac{t^2 + 1}{t} \right)^\frac{3}{2} \left( \frac{t^2-1}{t^2} \right) \)

Question 8: Find the solution of the differential equation:

\( (\frac{dy}{dx}) = \frac{\sec^2x}{\tan^2x} \)

Answer: \( (\frac{dy}{dx}) = \frac{\sec^2x}{\tan^2x} \)

\( dy = (\frac{dy}{dx}) dx \)

\[ = \left( \frac{\sec^2x}{\tan^2x} \right) dx \]

\[ = \left( \frac{1}{u^2} \right) du \] \( \because u = \tan x \) \( \Rightarrow \) \( du = \sec^2x dx \)

\( d = \left( \frac{1}{u} \right) \)

Else

\( y = \left( \frac{-1}{u} \right) + c \)

\[ = - \left( \frac{1}{\tan x} \right) + c \]

\( x + c = -\cot \)

verification: \( (\frac{d}{dx}) (-\cot x + c) = \cot^2x + 1 = \left( \frac{1 + \tan^2x}{\tan^2x} \right) = \left( \frac{\sec^2x}{\tan^2x} \right) \)

Question 9: Find the solution of the differential equation determined by the initial conditions:

\( (\frac{dy}{dx}) = x^2-2x-4; \) initial conditions: \( y = -6 \) when \( x = 3 \)

Answer:

\( dy = (\frac{dy}{dx}) dx \)

\[ = (x^2-2x-4) dx \]

\[ = d \left( \frac{x^3}{3} - x^2 - 4x \right) \]

Else

\( y = \left( \frac{x^3}{3} \right) - x^2 - 4x + c \)
substituting the initial condition

\[-6 = (\frac{(3)^3}{3}) - (3)^2 \cdot 4 \cdot (3) = 6 + c \Rightarrow c\]

Else

\[y = \frac{(X^3)}{3} + 4x - x^2 - 6\]

**Activity 2 - Numerical series**

**Introduction**

This activity presents the:

- Riemann integral, it is a small digression summations, infinite sums and numerical series.
- Series nature of the tests.
- Series of positive terms.
- Alternating series.
- Absolute and conditional convergence.
- Other convergence test.

**Activity Detail**

The sum of \( n \) successive terms in a sequence is designated by the symbol sum

\[\Sigma_{k=1}^{n} a_k\]

where the change rate of the sum, \( k \) indicates the position of the portion of the sum and takes values in \( \mathbb{N} \), beginning with the value indicated by low sum and end symbol in the indicated value on top of this symbol, the summation of the following properties are shown from the sum of the properties:

\[\Sigma_{k=1}^{n} (\lambda a_k + \beta b_k) = \lambda \Sigma_{k=1}^{n} a_k + \beta \Sigma_{k=1}^{n} b_k\] linearity

\[\Sigma_{k=p}^{n} (a_k - a_{kp}) = (a_n + \cdots + a_{n-p+1}) - (a_0 + \cdots + a_{p-1})\] \( p \) lag terms

\[\Sigma_{k=1}^{n} a_k \cdot \Sigma_{k=1}^{n} b_k = \Sigma_{k=1}^{n} (a_k b_k + a_{k-1} b_2 + \cdots + a_1 b_n)\] multiplication

Note: Replacing the summation index is so frequent, that is, using the same change index, provided that the value and the order of terms to add remain invariant.

to any \( n \in \mathbb{N} \) que:

\[\Sigma_{k=1}^{n} k = \left(\frac{n(n+1)}{2}\right)\]

\[\Sigma_{k=1}^{n} k^2 = \left(\frac{n(n+1)(2n+1)}{6}\right)\]

\[\Sigma_{k=1}^{n} k^3 = \left(\frac{n^2(n+1)^2}{4}\right)\]

\[\Sigma_{k=1}^{n} k^4 = \left(\frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}\right)\]
A statement is made as follows:

(1) It has to be

\[ \sum_{k=1}^{n} [k + (n- (k-1))] = \sum_{k=1}^{n} (n + 1) = n (n + 1) \]

But

\[ \sum_{k=1}^{n} (n- (k-1)) = \sum_{k=1}^{n} k \]

adding back to the front. Therefore

\[ \sum_{k=1}^{n} (n + 1) = 2 \sum_{k=1}^{n} k \]

and the result follows induction:

(2) demonstration of

For \( n = 1 \) one has \( \sum_{k=1}^{1} k^2 = 1 = (1 (1 +

Assuming a valid expression for \( n \)

\[ \sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n + 1)^2 \]

\[ = (n (n + 1) (2n + 1)) / 6 + (n + 1)^2 \]

\[ = (((n + 1) (2n^2 + 7n + 6)) / 6) \]

\[ = (((n + 1) ((n + 1) +1) (2 (n + 1) +1)) / 6) \]

(3) and (4) also proves induction.

**Endless Numerical Series and Sums**

Remember a few facts about sequences of real numbers

A sequence is a function \( (x_\{n\}) : \mathbb{N} \rightarrow \mathbb{R} \), the set of its terms is represented by \( \{x_\{n\}\} \) and its general term is \( x_\{n\} \).

Nature (convergent or not) of a sequence \( (x_\{n\}) \) is determined by the behavior of its “tail”, not being affected by any finite initial part:

to \( \forall k \in \mathbb{N} \) if \( \lim_{n \rightarrow \infty} x_\{n+k\} = L \) then \( \lim_{n \rightarrow \infty} x_\{n\} = L \)

Some notable sequences studied previously, are listed with \( n \in \mathbb{N} \) index:

\( x_\{n\} = r^n \) (arithmetically), where \( r \) is a constant

\( x_\{n\} = r^n \) (geometric progression ratio \( r \))

\( x_\{n\} = (1 / n) \) (harmonic)

\( x_\{n\} = (1 / (n ^ (p))) \) (p-harmonic)

\( x_\{n\} = (1 / (n!)) \) (exponential)

\( x_\{n\} = u_\{n\} - u_\{n + p\} \) (Mengoli)
The sum (infinite) of all the terms of a sequence \((x_\{n\})\) is called a number (infinite) formal

\[
\sum_{k = 1}^{\infty} x_\{k\} = x_1 + x_2 + \cdots + x_3
\]

to determine when the sum of an infinite number of parcels studied in the sequence of partial sums of a sequence.

The cumulative sum of the terms of the given sequence \((x_\{n\})\) is itself a sequence of partial sums \((s_\{n\})\) whose general term is given by

\[
s_\{n\} = x_1 + x_2 + \cdots + x_\{n\} = \sum_{k = 1}^{\{k\}} x_\{k\}
\]

The limit of the partial sums, if any, is defined as the sum of the series and the series is said to be converged,

\[
\sum_{k = 1}^{\{k\}} x_\{k\} = \lim_{n \to \infty} \sum_{k = 1}^{\{k\}} x_\{k\}
\]

When the limit does not exist the series is said to be divergent. Reciprocally, the difference (later) between consecutive terms a sequence \((s_\{n\})\) results in an associated sequence \((x_\{n\})\) whose terms are recursively given by

\[
x_\{n\} = \Delta s_\{n\} = s_\{n\} - s_\{n-1\} \text{ to } n \in \mathbb{N}
\]

\(x_0\) is arbitrary

in a sense, the \(\sum\) and \(\Delta\) operators are inverses of each other, for

Example 1: The sequence given by \(x_\{\text{general term } \{n\}\} = (1 / (2^n))\) determines a sequence of partial sums

\[s_\{n\} = \sum_{k = 1}^{\{k\}} (1 / (2^\{k\}))\]

The limit exists in \(\mathbb{R}\)

\[
\lim_{n \to \infty} \sum_{k = 1}^{\{k\}} (1 / (2^\{k\})) = 1
\]

Example 2: The sequence given by \(x_\{\text{general term } \{n\}\} = (1 / n)\) determines a sequence of partial sums

\[s_\{n\} = \sum_{k = 1}^{\{k\}} (1 / k)\]

The limit does not exist in \(\mathbb{R}\) (will be shown below)

\[
\lim_{n \to \infty} \sum_{k = 1}^{\{k\}} (1 / k) = \infty
\]

Why infinite sum plots exist in the first case and not in the second?
Conditions of Convergence

Condition is necessary, if the $\sum_k$ series $\{k = 1\} \lim_\infty x_\{k\}$ is convergent, then $\lim_\{n \to \infty\} x_\{n\} = 0$

For the demonstration, let $s_\{n\} = \Sigma_\{k = 1\} x_\{k\}$ for hypothesis

$\lim_\{n \to \infty\} s_\{n\} = S$ follows that $\forall \varepsilon > 0$

$\exists N: n > N \Rightarrow |s_\{n\} - s| < (\varepsilon / 2)$

$\exists N: n > N \Rightarrow |s_\{n + 1\} - s| < (\varepsilon / 2)$

Logo

$|x_\{n + 1\}| = |S_\{n + 1\} - s_\{n\}|$

$= |s_\{n + 1\} - s_\{n\}| + ss$

$= |s_\{n + 1\} - s_\{n\}| - (s_\{n\} - s )$

$| \leq |s_\{n + 1\} - s_\{n\}| + |s_\{n\} - s |$

$< (\varepsilon / 2) + (\varepsilon / 2) = \varepsilon$

Proving

$\lim_\{n \to \infty\} x_\{n\} = 0$

The $\lim_\{n \to \infty\} x_\{n \to \infty\}$ condition $x_\{n \to \infty\} = 0$ is necessary for the convergence $\sum_k {k = 1} \lim_\infty x_\{k\}$, but it is not sufficient as shown the two examples above:

In the first example $\{1 / (2^n)\} \to 0$ and $\sum_k {k = 1} \lim_\infty (1 / (2^k)) = 1$

but in the second example $\{1 / n\} \to 0$ but $\sum_k {k = 1} \lim_\infty (1 / k)$ diverges

(sufficient condition of divergence)

the reciprocal against the previous proposition establishes a sufficient condition of divergence:

If $\lim_\{n \to \infty\} x_\{n\} \neq 0$ then $\sum_k {k = 1} \lim_\infty x_\{k\}$ is divergent.

A series converges if the sequence of partial sums is convergent, therefore, the same sequences convergence conditions may be used to determine the convergence conditions of a sequence of partial sums.
(necessary and sufficient condition for convergence) a series $\Sigma_\{k = 1\}^{\infty} x_\{k\}$ converges in $\mathbb{R}$, iff the sequence of partial sums $(s_\{n\})$ are a Cauchy sequence

$$\forall \varepsilon > 0, \exists N: m, n > N$$

$$\Rightarrow |s_\{m\} - s_\{n\}| < \varepsilon$$

$$\Leftrightarrow |x_\{m + 1\} + x_\{m + 2\} + \cdots + x_\{n\}| < \varepsilon$$

In demonstration, check that any convergent sequence is Cauchy and how $\mathbb{R}$ is complete, so any sequence Cauchy converges.

for counter-reciprocal if $(s_\{n\})$ is not Cauchy, then $\Sigma_\{k = 1\}^{\infty} x_\{k\}$ does not converge.

**Remarkable Series**

The partial sums of the terms of the notable sequences results in the following sequences, listed with the change index $k = 1, \ldots$,

- arithmetic, constant $r$: $s_\{n\} = \{k \Sigma_\{n\}^\{kr\} = 1\} = ((n(n + 1)) / 2)r$
- geometric $s_\{n\} = \{\Sigma_\{n\}^\{r\} k = 1\} = ((r- r^{n+1}) / (1- r))$
- harmonic: $s_\{n\} = \Sigma_\{k = 1\}^n (1 / k)$
- p-harmonic: $s_\{n\} = \Sigma_\{k = 1\}^n (1 / (k^p))$
- exponential: $s_\{n\} = \Sigma_\{k = 1\}^n (1 / (k!))$

Mengoli (difference of a sequence with a gap $p$ her own): $\Sigma_\{k = 1\}^n a_\{k\} = \Sigma_\{k = 1\}^n (u_\{k\} - u_\{k\} + p)$

What is the limits of these sequences Partial sums and in what cases are there?

arithmetic series $\Sigma_\{k = 1\}^{\infty} kr$ converges only if $r = 0$, because $kr$ sequence diverges for any

the harmonic series $\Sigma_\{k = 1\}^{\infty} (1 / k)$ is divergent.

To prove it, consider the difference of $s_\{n\}$ subsequence $(2n) = \Sigma_\{k = 1\}^{2n} (1 / k)$ with $s_\{n\}$

$$s_\{2n\} = -s_\{n\} \Sigma_\{k = 1\}^{2n} (1 / k) - \Sigma_\{k = n + 1\}^{2n} (1 / k) = (\Sigma_\{n + 1\}^{n + 1} (1 / k) + (1 / (n + 1)) + \cdots + (1 / (n + n)) > (1 / (2n)) + (1 / (2n)) + \cdots + (1 / (2n)) = (n / (2n)) = (1/2)$$

Then $\forall n, |s_\{2n\} - s_\{n\}| > (1/2)$

Logo $(s_\{n\})$ cannot be Cauchy, which proves that is not converging$(\infty)$.

the geometric series $\Sigma_\{k = 1\}^{r} (r \wedge k)$ converges to $(r / (1-r))$ if $|r| < 1$ and diverges otherwise to prove, let

$$s_\{n\} = \Sigma_\{k = 1\}^{n} r$$
record himself to equality have proved by induction

\[(1-r^n) = (1-r)(1 + r + r^2 + r^3 + \cdots + r^{n-1})\]

manipulating a little get up

\[s_n = \sum_{k=1}^{n} r^k = \frac{(r-r^{n+1})}{(1-r)}\]

Therefore if \(|r| < 1\) then \(\lim_{n \to \infty} s_n = 0\) \(r^{n+1}\) logo

\[\lim_{n \to \infty} s_n = \frac{(r-r^{n+1})}{(1-r)} = \frac{r}{1-r}\]

and \(|r| \geq 1\) the limit does not exist\(\sum a_n\).

series of Mengoli, \(\sum_{k=1}^{n} a_k\) is convergent if \(\lim_{n \to \infty} u_n = L\) and divergent if \(\lim_{n \to \infty} u_n = \pm \infty\). Further, the sum is given by

\[\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (u_k - u_{k+p}) = (u_1 + \cdots + u_p) - (u_{n+1} + \cdots + u_{n+p})\]

do prove if \(\lim_{n \to \infty} u_n = L\) then \(\lim_{n \to \infty} u_{n+p} = L\) for any \(p\)

So \(\sum_{k=1}^{n} a_k\) is Cauchy., then converges

For the property “telescoping” only the initial terms \(p\) and \(p\) final terms remain in sum:

\[\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (u_k - u_{k+p}) = (u_1 + \cdots + u_p) + \sum_{k=1}^{n} (u_{k+1} + \cdots + u_{n+p})\]

and the limit follows.

Example 1: Determine the nature of \(\sum_{n=1}^{\infty} (1 / n (n + 3))\) and is converged calculate the sum.

In partial fraction, we obtain

\[\sum_{k=1}^{n} (1 / (n(n + 3))) = \left(\sum_{k=1}^{n} (1 / n) - (1 / ((n + 3)))\right)\]

As \(\lim_{n \to \infty} (1 / n) \Sigma = 0\) then \(\sum_{n=1}^{\infty} (1 / n (n + 3)))\) converges

The sum is obtained

\[\sum_{n=1}^{\infty} (1 / n) - (1 / ((n + 3))) = 1+\]

**Nature Series**

A series of convergence or divergence is the nature. The nature of the series can be determined by comparing them with the notable series, the nature which has already been determined.

\(\sum_{k=1}^{\infty} a_k\) and \(\sum_{n=1}^{\infty} b_n\) series \(u_k\) and \(v_k\) that differ only in a finite number of terms are of the same nature (or both diverge or both converge).

The statement follows a similar proposition on sequences applied to \(U_n = \sum_{k=1}^{n} a_k\) and \(V_n = \sum_{k=1}^{n} b_k\), which are also sequences (partial sums series);

The sum or multiplication of a number by a constant does not change the nature of the linear combination of any series with a convergent series and does not alter its original nature
For example

\[ \Sigma_{k = 1}^{\infty} \frac{w}{k} \] is divergent for any \( c \in \mathbb{R} \) because

\[ \Sigma_{k = 1}^{\infty} \frac{w}{k} = c \Sigma_{k = 1}^{\infty} \frac{1}{k} \]

and \( \Sigma_{k = 1}^{\infty} \frac{1}{k} \) is divergent.

**Positive terms Series, Comparison Tests**

A series of positive terms \( \Sigma_{k = 1}^{\infty} a_k \), as the name implies, the plots are positive: \( a_k \geq 0, \forall k \in \mathbb{N} \) terms.

A number of positive converges if its sequence partial sums is increased.

Example:

For demonstrating note that a sequence of partial positive terms sums is increasing, then converges if is increased.

The sequence of Partial sum of exponential series can be compared with the sequence Partial sums of geometric series and as \( (1 / (k!)) \leq (1 / (2^{k-1})) \) for any \( k \), follows that

\[ \Sigma_{k = 1}^{\infty} \frac{1}{(k!)} \leq \Sigma_{k = 1}^{\infty} \frac{1}{(2^{k-1})} < 2 \]

Logo \( \Sigma_{k = 1}^{\infty} \frac{1}{(k!)} \) is convergent.

The nature of a series of positive terms can be determined by comparison with positive terms series whose nature is known.

(Comparison test) Be \( \Sigma_{k = 1}^{\infty} u_k \) a series of positive terms,

If \( \Sigma_{k = 1}^{\infty} u_k \) is a convergent series (with positive terms) and eventually \( u_k \leq v_k \) then \( \Sigma_{k = 1}^{\infty} u_k \) is convergent;

If \( \Sigma_{k = 1}^{\infty} w_k \) is a divergent series (with positive terms) and eventually \( w_k \leq u_k \) then \( \Sigma_{k = 1}^{\infty} u_k \) is divergent;

Example:

Determine the nature of \( \Sigma_{k = 1}^{\infty} \frac{1}{\sqrt{k}} \)

As \( (1 / n) \leq (1 / (\sqrt{n})) \) for any \( n \in \mathbb{N} \) and \( \Sigma_{k = 1}^{\infty} \frac{1}{k} \) is divergent, then \( \Sigma_{k = 1}^{\infty} \frac{1}{\sqrt{k}} \) is also divergent.

The determination of nature can also be made by comparing the order of convergence of its associated sequences \( u_n \) and \( v_n \).

(Comparison of limits) are \( \Sigma_{k = 1}^{\infty} u_k \) and \( \Sigma_{k = 1}^{\infty} v_k \) two series of positive terms.

If \( \lim_{n \to \infty} \frac{u_n}{v_n} = c > 0 \), ie if \( u \in O(v) \), then the two series have the same nature;

If \( \lim_{n \to \infty} \frac{u_n}{v_n} = 0 \), ie if \( u \in o(v) \), then \( \Sigma_{k = 1}^{\infty} v_k \) converges, \( \Sigma_{k = 1}^{\infty} u_k \) converges;
If \( \lim_{n \to \infty} \frac{(u_n)}{(v_n)} = +\infty \), i.e. if \((u_n) \in \omega(v_n)\), then \(\sum_{k=1}^{\infty} v_k\) diverges, \(\sum_{k=1}^{\infty} u_k\) diverges;

Example

The \(p\)-harmonic series \(\sum_{k=1}^{\infty} \frac{1}{k^p}\) diverges and converges to \(p \leq 1\) \(p > 1\)

If \(p = 1\) the series is harmonic and diverges as seen.

If \(p < 1\) then \(n \geq n^p\) \(\Rightarrow\) \((1/n) \leq (1/(n^p))\) as soon as \(\sum_{k=1}^{\infty} \frac{1}{k^p}\) is divergent, \(\sum_{k=1}^{\infty} \frac{1}{k^p}\) is also divergent.

If \(p > 1\) the groups is as follows series

\[
\sum_{k=1}^{\infty} \left(\frac{1}{k^p}\right) = (\frac{1}{1^p}) + (\frac{1}{2^p}) + (\frac{1}{3^p}) + \ldots
\]

and compares with the series

\[
\sum_{k=1}^{\infty} \left(\frac{1}{(2^k)^p}\right) = (\frac{1}{1^p}) + (\frac{1}{2^p}) + (\frac{1}{4^p}) + \ldots
\]

It appears that

\[
\sum_{k=1}^{\infty} \left(\frac{1}{k^p}\right) \leq \sum_{k=1}^{\infty} \left(\frac{1}{(2^k)^p}\right) = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{(2^k)^p}\right)
\]

\[
= 1 + \sum_{k=1}^{\infty} \frac{1}{(2^k)^p} = 1 + \frac{1}{1 - 2^p} = \frac{2 - 2^p}{1 - 2^p}
\]

Soon \(\sum_{k=1}^{\infty} \frac{1}{k^p}\) is convergent for \(p > 1\).

A series of positive terms is convergent, the rearrangement of its terms does not change its nature.

This is not the case in certain alternating series as will be seen further ahead!

**Alternating Series, Absolute and Conditional Convergence**

\(\sum_{series\{k=1\}}^{\infty} (-1)^{k} a_k\) or \(\sum_{k=1}^{\infty} (-1)^{k+1} a_k\) where \(a_k > 0\) for any \(n \in \mathbb{N}\) are called alternating series.

(Test of alternating series) In alternating series \(\sum_{k=1}^{\infty} (-1)^{k+1} a_k\), will eventually \(a_{n+1} \leq a_n\) and \(\lim_{n \to \infty} a_n = 0\), then the alternating series converges.

For the demonstration, in \(s_1\) adding all terms (finite number) such that \(a_k \leq 0\) and \(a_{k+1} > a_k\), renumber the subsequent Partial sums so that \(a_n > 0\) and \(a_{n+1} \leq a_n\) to \(n \in \mathbb{N}\).
By grouping the terms in a way you obtain \( s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \)

and as \( a_{n+1} < a_n \) and then \( n \in \mathbb{N} \) so on \( s_{2n} < s_{2n+2} < \cdots < s_{2n} \) is increasing

Grouping the terms otherwise \( s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \)

is obtained that \( s_{2n} < a_1 \) to any \( n \in \mathbb{Z} \) so on \( s_{2n} \) is limited.

It follows that the sequence \( \{ s_n \} \) is limited. Let \( \lim_{n \to \infty} s_{2n} = S \leq a_1 \)

As \( s_{2n+1} = S - a_{2n+1} \)

So \( \lim_{n \to \infty} s_{2n+1} = \lim_{n \to \infty} s_{2n} = S \) by hypothesis \( \{ n \to \infty \} a_n = 0 \)

Then for any \( \varepsilon > 0 \), there \( N_1 > 0 \) such that \( 2n > N_1 \) \( \Rightarrow \) \( s_{2n} < \varepsilon \)

And also there \( N_2 > 0 \) such that \( 2n + 1 > N_2 \) \( \Rightarrow \) \( s_{2n+1} < \varepsilon \)

\( N > \max \{ N_1, N_2 \} \) then \( n > N \) it follows that \( \lim_{n \to \infty} \{ s_n \} = S < \varepsilon \)

Logo \( \lim_{n \to \infty} s_n = S \).

Example:

The \( \sum \) series \( \{ k = 1 \} \) is divergent because \( \lim_{n \to \infty} 1 \neq 0 \).

Although the sum of the series, when grouped in a way be null!

\[
\sum_{k=1}^{\infty} (-1)^{k} \rightarrow 1 + (-1 + 1) + (-1 + 1) + \cdots = 0 + 0 + \cdots = 0
\]

But, when the sum of the number of grouped differently, have another value!

\[
\sum_{k=1}^{\infty} (-1)^{k} \rightarrow (-1)^{k} \rightarrow -1 + (1-1) + (1-1) + \cdots = -1 + 0 + 0 + \cdots = -1
\]

The limit of the sequence of partial sums does not exist because oscillates

\[
\lim_{n \to \infty} \sum_{k=1}^{n} (-1)^{k} \text{ does not exist}
\]

An infinite series \( \sum_{k=1}^{\infty} a_k \) is absolutely convergent iff the \( \sum \) series \( \{ k = 1 \} \) converges to \( a_k \) is convergent.

A convergent series but not absolutely convergent is said to be conditionally convergent.

Example:

The \( \sum \) series \( \{ k = 1 \} \) converges for \( 0 < (1 / (k + 1)) < (1 / k) \) to any \( k \in \mathbb{N} \) and \( \lim_{n \to \infty} (1 / n) = 0 \).
But it is not absolutely convergent because \( \sum_{k=1}^{\infty} \frac{1}{k} \) is divergent.

Soon she is conditionally convergent.

(Theorem of Riemann) If the series is Conditionally convergent Then,

by a rearrangement of terms in the series, the sum may be set to any arbitrary value, and further, can be made up divergent or oscillating.

For the demonstration, let \( x_\{p\} \) be the sum of the first \( p \) positive terms and \( -y_\{n\} \) be the sum of the first \( n \) negative terms.

For \( p \to \infty \) en follows that

\[
\lim (x_\{p\} - y_\{n\}) = \lim (x_\{p\} + y_\{n\}) = \infty
\]

Soon:

\[
\lim_{p \to \infty} x_\{p\} = \infty \text{ and } \lim_{n \to \infty} y_\{n\} = \infty
\]

To adjust the sum of the series for a \( \sigma \) value, choose the lesser \( p_1 \) to have \( x_\{p\} > \sigma \) . Just as you can choose the lowest \( n_1 \) to have \( y_\{n\} < \sigma \). In the rearrangement put up the primieros \( p_1 \) positive terms in their original order, followed by \( n_1 \) negative terms, also in the original order.

If \( S_\{v\} \) is the sum of \( v \) terms, then

\[
S_\{v\} < \sigma \text{ if } v < p_1
\]

but

\[
S_\{v\} > \sigma \text{ is } p_1 \leq v < p_1 + n_1
\]

the process is continued by placing the third group \( (P_2 - p_1) \) positive terms where \( P_2 \) is the smallest index such that \( x_\{P_2\} > y_\{n_1\} + \sigma \) and the fourth group \( (n_2 - n_1) \) negative terms where \( n_2 \) is the smallest index such that \( y_\{n_2\} > x_\{p_2\} - \sigma \).

Continuing in this way, it is clear that \( p \{ r \} + n_\{r\} > v \geq \sigma \{ r \} \) and the take \( p \{ r \} \) values \( n_\{r\} \) so as to be the first to meet indices

\[
\begin{align*}
x_\{p_1\} > \sigma_1 \\
y_\{n_1\} > x_\{p_1\} - \sigma_1 \\
x_\{p_2\} > y_\{n_1\} + \sigma_2 \\
y_\{n_1\} > x_\{p_2\} - \sigma_2
\end{align*}
\]

\[
\lim S_\{v\} = \sigma
\]

To make divergent or oscillating amount, a divergent sequence is taken or oscillate \( \sigma_\{r\} \) and the take \(-p_1\) values \( n_1 , \ldots \) so as to be the first to meet indices
and so on.

If a $\Sigma_-$ series \( \{ k = 1 \} ^{\infty} a_- \{ k \} \) is absolutely convergent then it is convergent

\[
| \Sigma_- \{ k = 1 \} ^{\infty} a_- \{ k \} | \leq | \Sigma_- \{ k = 1 \} ^{\infty} | a_- \{ k \} |
\]

In the test, as $| a_- \{ k \} |$ equals $a_- \{ k \}$ or $-a_- \{ k \}$ has any $k \in \mathbb{N}$ that

\[
0 \leq a_- \{ k \} + | a_- \{ k \} | \leq 2 | a_- \{ k \} |
\]

So

\[
0 \leq \Sigma_- \{ k = 1 \} ^{\infty} ( a_- \{ k \} + | a_- \{ n \} | ) \leq 2 \Sigma_- \{ k = 1 \} ^{\infty} | a_- \{ k \} |
\]

By hypothesis and by comparing test results that $\Sigma_- \{ k = 1 \} ^{\infty} ( a_- \{ k \} + | a_- \{ n \} | )$ is convergent.

Soon $\Sigma_- \{ k = 1 \} ^{\infty} a_- \{ k \} = \Sigma_- \{ k = 1 \} ^{\infty} ( a_- \{ k \} + | a_- \{ n \} | ) \Sigma_- \{ k = 1 \} ^{\infty} | a_- \{ k \} |$ It is convergent, proving the first part.

As $S = \Sigma_- \{ k = 1 \} ^{\infty} a_- \{ k \} \leq \Sigma_- \{ k = 1 \} ^{\infty} | a_- \{ k \} | = L$

logo $S \leq L$

Also follows that $S = -\Sigma_- \{ k = 1 \} ^{\infty} a_- \{ k \} \leq \Sigma_- \{ k = 1 \} ^{\infty} | a_- \{ k \} | = L$

logo $-S \leq L$

Of the two inequalities, $S \leq L$ and $S \geq -L$ follows that $| S | \leq L$

**Other convergence tests**

The ratio test is useful or D’ Alembert to determine the nature series where the index appears as factor as well as to determine the absolute convergence of a series:

Be $\Sigma_- \{ k = 1 \} ^{\infty} u_- \{ k \}$ such that a series $u_- \{ k \} \neq 0$, $\forall k \in \mathbb{N}$.

If $\lim_{n \to \infty} | ( u_- \{ n + 1 \} ) / ( u_- \{ n \} ) | = L < 1$ the series is absolutely convergent;

If $\lim_{n \to \infty} | ( u_- \{ n + 1 \} ) / ( u_- \{ n \} ) | = L > 1$ or else $\lim_{n \to \infty} | ( u_- \{ n + 1 \} ) / ( u_- \{ n \} ) | = + \infty$, the series diverges;

If $\lim_{n \to \infty} | ( u_- \{ n + 1 \} ) / ( u_- \{ n \} ) | = 1$, the test is inconclusive;

In the demonstration, let $r$ such that $\lim_{n \to \infty} | ( u_- \{ n + 1 \} ) / ( u_- \{ n \} ) | = L \leq r < 1$

Eventually, or from some index $N$, you have

\[
| ( u_- \{ n + 1 \} ) / ( u_- \{ n \} ) | < r \text{ for } n > N
\]

Then for successive values $N, N + 1, ...$ one has

\[
| U_- \{ N + 1 \} | < | U_- \{ N \} | r
\]

\[
| U_- \{ n + 2 \} | < | U_- \{ N + 1 \} | r < | U_- \{ N \} | r^2
\]
... 

\[ |U_{n+k}| < |U_{N}| r^k, \forall k \geq 1 \]

Follows that \( \Sigma_\{k=1\}^\infty |u_{k}| N |l < \Sigma_\{k=1\}^\infty |u_{k}| l r^k \)

And as the series \( \Sigma_\{k=1\}^\infty |u_{k}| l r^k \) is geometric with \( |r| < 1 \), then is convergent, then \( \Sigma_\{k=1\}^\infty |u_{n+k}| = \Sigma_\{k=N+1\}^\infty |u_{k}| l \) It is convergent.

We conclude that \( \Sigma_\{k=1\}^\infty |u_{k}| l \) is also convergent for a finite number of shares does not change the nature of a series.

In the second case, if \( \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = L > 1 \), then there is \( N \) such that

\[ \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| > 1 \Rightarrow |u_{n+1}| > |u_n| \text{ for } n > N \]

Soon \( \lim_{n \to \infty} u_n \neq 0 \) and \( \Sigma_\{k=1\}^\infty |u_{k}| l \) diverges.

Ex: Applying the ratio test the \( \Sigma_\{n=1\}^\infty \left( (-1)^n \left( \frac{n^3}{3^n} \right) \right) \), we obtain

\[ \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{3^n (n+1)^3}{3^{n+1} n^3} \right| \]

\[ = \left| \frac{3^n (n+1)^3}{3^{n+1} n^3} \right| = \left( \frac{1}{3} \right)^2 \frac{(n+1)^3}{n^3} = \frac{1}{3} (1+(1/n))^3 \]

Soon the series is absolutely convergent and then converged.

The pattern of root or Cauchy is useful for determining the nature series where the index appears in the exponent and for determining the absolute convergence of a series:

For \( \Sigma_\{k=1\}^\infty |u_{k}| l \)

If \( \lim_{n \to \infty} [n] \sqrt[n]{|u_n|} = L < 1 \) the series is absolutely convergent;

If \( \lim_{n \to \infty} [n] \sqrt[n]{|u_n|} = L > 1 \) or else \( \lim_{n \to \infty} [n] \sqrt[n]{|u_n|} = + \infty \), the series diverges;

If \( \lim_{n \to \infty} [n] \sqrt[n]{|u_n|} = 1 \) the test is inconclusive;

(Dirichlet Criterion) Be \( \Sigma_\{k=1\}^\infty |u_{k}| l \) such that a series \( u_{k} \geq 0, \forall k \in \mathbb{N} \). We may figure out if there

\[ \lambda = \lim_{n \to \infty} [n] \lambda \left( |u_{k}| l \right) \]

If \( \lambda = 0 \) with \( \alpha > 1 \) the series converges;

If \( \lambda = 0, + \infty \) with \( \alpha > 1 \) the series converges;

If \( \lambda = + \infty \), with \( \alpha \leq 1 \) the series diverges;

If \( \lambda = 0, + \infty \) with \( \alpha \leq 1 \) the series diverges;

The test results from the comparison with the p-harmonic series \( \Sigma_\{k=1\}^\infty \left( \frac{1}{n^\alpha} \right) \)
Summary of Convergence Tests

Given a series \( \sum_{k = 1}^{\infty} u_k \), how to decide efficiently, which test to apply?

Check immediately that \( \lim_{k \to \infty} u_k \neq 0 \) the series is divergent, otherwise it is necessary to proceed with the resolution.

Identify one of the remarkable series. Some algebraic manipulation may be required.

A series can be decomposed in a difference in terms of a single sequence offset of a fixed value \( k + p \), then it is a series of Mengoli and can be resolved.

If it is a remarkable series but have a similar shape, so direct comparison tests or limits may be the most appropriate.

If the series is alternating, verify that it is dull and \( \lim_{k \to \infty} u_k = 0 \) then converges.

If the series involves factorial and exponents of the index, then the ratio test can result.

If the series involves exponents of the index, then the root of the test may result.

Conclusion

In order to present the Riemann integral, there was a small digression summations, infinite sums and numerical series. Series nature of the tests. Series of positive terms. Alternating series, absolute and conditional convergence. Other convergence tests.

Evaluation

Group a minimum of 5 and a maximum of 10 questions in each of the following 1 hour evaluation test without reference.

Choose different type questions in the preparation of each test, the changing parameters and database to prevent bonding.

Each question is worth between 10% and 20% depending on the number of selected questions.

Question 1: Find the five following elements of partial sums \( s_n \) and get a formula for the general term \( s_n \). Determine also the infinite series is convergent or divergent; if convergent determine their sum: \( \sum_{n = 1}^{\infty} \left( \frac{2}{5^{n-1}} \right) \)

Answer:

\[
\sum_{n = 1}^{\infty} \frac{2}{5^{n-1}} = 2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \frac{2}{625} = \frac{1562}{625} = 2.4992
\]

\[
s_n = \sum_{i = 1}^{n} \left( \frac{2}{5^{i-1}} \right)
\]

\[
2\sum_{i = 1}^{\infty} \left( \frac{1}{5} \right)^i = \frac{2}{1 - \frac{1}{5}}
\]

\[
2\sum_{i = 0}^{\infty} \left( \frac{1}{5} \right)^i = 2 + 2\sum_{i=1}^{\infty} \frac{1}{(1/5)^i}
\]
= 2+2(((1/5)-((1/5))ⁿ⁺¹) / (1-(1/5)))

= 2+(((1-((1/5))ⁿ)/2))

Calculating the limit

\[ \lim_{n \to \infty} \{ (2 + ((1 - ((1/5)^n)) / 2)) \} = 2 + (1/2) = (3/2) \]

Question 2: Find the five following elements of partial sums \( s_\{n\} \) and get a formula for the general term \( s_\{n\} \). Determine also the infinite series is convergent or divergent; Convergent determine if the sum:

\[ \sum_{n = 1}^{\infty} \frac{2n + 1}{n^2 (n + 1)^2} \]

Answer:

\[ \Sigma_{n = 1}^{5} \frac{(2n + 1)}{(n^2 (n + 1)^2)} = (3/4) + (5 / (4 \cdot 9)) + (7 / (9 \cdot 16)) + (9 / (16 \cdot 25)) + ((11) / (25 \cdot 36)) = (35) / (36) \]

\[ s_\{n\} = (\Sigma_{p = 1}^{n} \frac{(2p + p)}{(p^2 (p + 1)^2)} ) \]

How is a fraction of two polynomials in the numerator of degree less than the degree of the denominator, are sought partial fractions

\[ \frac{(2p + 1)}{(p^2 (p + 1)^2)} = \frac{A}{p^2} + \frac{B}{(p + 1)^2} \]

Comparing numerator coefficients on both sides:

\[ A = 1 \]
\[ 2A = 2 \]
\[ = A + B = -1 \iff B \]

\[ ((2p+1)/(p^2(p+1)^2))=(1/(p^2))-(1/((p+1)^2)) \]

Logo

\[ s_\{n\} = \sum_{p = 1}^{\infty} \frac{(2p + p)}{(p^2 (p + 1)^2)} \]

\[ = \sum_{p = 1}^{\infty} \frac{(1/p^2)}{(1/((p+1)^2))} \]

\[ = 1 - (1/((n+2)^2)) \]

Calculating the limit

\[ \lim_{n \to \infty} (1 - (1/((n+2)^2))) = 1 \]

Question 3: Find the five following elements of partial sums \( s_\{n\} \) and get a formula for the general term \( s_\{n\} \). Determine also the infinite series is convergent or divergent; if convergent determine their sum:

\[ \sum_{n = 1}^{\infty} \cos \pi n \]
Answer:

\[ \Sigma \_ { n = 1} ^ { 5} \cos n = -1 + 1 - 1 + 1 - 1 = -1 \]

\[ s_\{ r \} = \Sigma \_ { n = 1} ^ { r} \cos n \]

\[ \Sigma \_ { n = 1} ^ { r} \cos n = (-1)^n \]

\[ = \left( (-1) \left( \frac{R}{1} \right) - 1 \right) / 2 \]

Calculating the limit

\[ \lim_{r \to \infty} \left( (-1)^r - 1 \right) / 2 \]

\[ = \text{não existe} \]

\[ \therefore \text{oscila, logo divergente} \]

Question 4: Determine whether the series is convergent or divergent, convergent if the sum found: \( \Sigma \_ { n=1} ^ { \infty} \left( \frac{3}{2^n} \right) - \left( \frac{2}{3^n} \right) \)

Answer:

\[ \Sigma \_ { n=1} ^ { \infty} \left( \frac{3}{2^n} \right) - \left( \frac{2}{3^n} \right) = \left( \frac{1}{2} \right) - 2 \left( \frac{1}{3} \right) \]

\[ = 2 \]

\[ = \left( \frac{10}{3} \right) \]

Question 5: Determine whether the series is convergent or divergent, if convergent find their sum: \( \Sigma \_ { n=1} ^ { \infty} \left( \frac{\sin \left( \frac{4}{3} \pi \right) + 3}{4^n} \right) \)

Answer:

\[ \Sigma \_ { n=1} ^ { \infty} \left( \frac{\sin \left( \frac{4}{3} \pi \right) + 3}{4^n} \right) = \left( \frac{3}{4} \right) - \left( \frac{\sqrt{3}}{4} \right) \]

\[ = 1 - \left( \frac{1}{4} \right) \sqrt{3} \]

\[ = \left( \frac{10}{3} \right) \]

Question 6: Determine if the given series is convergent or divergent: \( \Sigma \_ { n=1} ^ { \infty} \left( \frac{\cos^2 n}{3^n} \right) \)

Answer:

\[ 0 \leq \left( \frac{\cos^2 n}{3^n} \right) \leq \left( \frac{1}{3^n} \right) \]

\[ 0 \leq \Sigma \_ { n=1} ^ { \infty} \left( \frac{\cos^2 n}{3^n} \right) \leq \Sigma \_ { n=1} ^ { \infty} \left( \frac{1}{3^n} \right) = \frac{1}{2} \]

Soon

\[ \Sigma \_ { n=1} ^ { \infty} \left( \frac{\cos^2 n}{3^n} \right) \text{ converges} \]

For the sequence of Parciás sums (positively) is increasing and bounded.

Question 7: Determine if the given series is convergent or divergent: \( \Sigma \_ { n=1} ^ { \infty} \left( \frac{3}{2n-\sqrt{n}} \right) \)
Answer:

\[0 \leq (3/2) \sum\{n = 1\} ^{\infty} (1 / n) \leq \sum\{n = 1\} ^{\infty} (3 / (2n-\sqrt{n}))\]

Soon

\[\sum\{1\} ^{n} = \{\infty\} (3 / (2n-\sqrt{n}))\]

By comparison with the harmonic series that diverges.

Question 9: Determine if the given series is convergent or divergent: \(\sum\{n = 1\} ^{\infty} (1 / (3^n - \cos n))\)

Answer:

\[0 \leq (1 / (\cos 3^n n)) \leq (1 / (2^n))\]

Converges, by comparison with the geometric series.

Question 10: Determine whether the alternating series is convergent or divergent: \(\sum\{n = 1\} ^{\infty} (-1) ^{n+1} (n / (2^n))\)

Answer: \(\sum\{n = 1\} ^{\infty} (-1) ^{n+1} (n / (2^n))\)

Converges as the general term is decreasing and infinitesimal

\[\lim\{n\} (n / (2^n)) = 0\]

also converges absolutely (and hence converges)

\[\lim\{n\} (1 / ((n + 1) \ln (n + 1))) = (1/2) < 1\]

Question 11: Determine whether the alternating series is convergent or divergent: \(\sum\{n = 1\} ^{\infty} (-1) ^{n+1} (1 / ((n + 1) \ln (n + 1)))\)

A. \(\sum\{1\} ^{\infty} (-1) ^{n+1} (1 / ((n + 1) \ln (n + 1)))\)

Converges as the general term is decreasing and infinitesimal

\[\lim\{1 / ((n + 2) \ln (n + 2))) = \lim\{1 / ((n + 1) \ln (n + 1)) = 0\]
Question 12: Determine whether the alternating series is convergent or divergent: \( \sum_{n=2}^{\infty} \frac{\sqrt{n+1}}{n-1} \)

A. \( \sum_{n=2}^{\infty} \frac{\sqrt{n+1}}{n-1} \) differs, the comparison test with the harmonic series
\[
\lim_{n \to \infty} \left( \frac{\sqrt{n+1}}{n-1} \right) = \frac{1}{n} \quad \text{lim}_{n \to \infty} \left( \frac{\sqrt{n}}{n-1} \right) = +\infty
\]

There is a series of Mengoli because although it can be separated into partial fractions, these are no gaps in the same sequence:
\[
\sum_{n=3}^{\infty} \frac{\sqrt{n}}{n-2} + \sum_{n=3}^{\infty} \frac{\sqrt{n}}{n+2}
\]

Partial fractions of this year were calculated as follows:
\[
\left( \frac{\sqrt{n}}{n-2} \right) = \frac{A}{\sqrt{n-2}} + \frac{B}{\sqrt{n+2}}
\]

\[
AB = 0 \quad A + B = 1
\]

\[ A = \frac{1}{2} = B \]

Question 13: Determine if the given series is absolutely convergent, conditionally convergent or divergent:
\[ \sum_{n=1}^{\infty} \frac{\cos \pi n}{n} \]

Answer: \( \sum_{n=1}^{\infty} \frac{\cos \pi n}{n} \) is conditionally convergent
\[ \sum_{n=1}^{\infty} \frac{\cos \pi n}{n} = \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{n} \right) \]

It is convergent since the general term is decreasing and infinitesimal
\[ \left( \frac{1}{n+1} \right) / \left( \frac{1}{n} \right) \leq 1 \]
\[ \lim_{n \to \infty} \left( \frac{1}{n} \right) = 0 \]

Question 14: Determine if the given series is absolutely convergent, conditionally convergent or divergent:
\[ \sum_{n=1}^{\infty} \frac{2n-1}{(3n-2)^2} \]

A. \( \sum_{n=1}^{\infty} \frac{2n-1}{(3n-2)^2} \) converges as compared with the second-harmonic series \( \sum_{n=1}^{\infty} \frac{1}{(n-2)^2} \) which converges.
\[ \left( \frac{2n-1}{(3n-2)^2} \right) = \left( \frac{1}{2 \cdot 3 \cdots 2n-1} \right) = \left( \frac{1}{2 \cdot 3 \cdots 2n-1} \right) \]
\[ = \left( \frac{1}{(2n-1)(3n-2)} \right) \leq \left( \frac{1}{n(n-2)} \right) \leq \left( \frac{1}{(n-2)^2} \right) \text{ for } n>2 \]
\[ \sum_{n=3}^{\infty} \frac{1}{(n-2)^2} \] converges.

\[ \sum_{n=1}^{\infty} \left( \frac{1}{(n-2)^2} \right) \] converges.
Activity 3 - Area closed regions and Fundamental Theorem of Calculus

Introduction

This activity provides the following:

Area closed regions calculation in the plan.

Definite integral and its properties.

Theorem of mean value for definite integrals.

Fundamental Theorem of Calculus.

Applications of definite integral: calculation of area of plane figures, volume of solids by cutting, circular disc and cylindrical housings, arc length, mass and time centroid of flatland.

Activity Details

Area Calculation

Remember the areas of the main regular plane figures (Triangular):

For decomposition squares or triangles can calculate the area of more complex figures:

![Complex figure](image)

The calculation of the area of plane figures is known as quadrature. The squaring of the circle is an old problem:

How to build a square, with only ruler and compass, which has the same area as a given circle

![Circle and square](image)

Geometric problems are solvable only with ruler and compass, equivalent to polynomials solutions with integer or rational coefficients.

In 1882, Lindemann showed that π is transcendental, ie is not polynomials or rational solution with integer coefficients, then the circle quadrature solution is an impossible problem.
The circle area can however be calculated, as shown by Archimedes, as a boundary of successive approximations.

The limits of calculation to determine the area of any plane figure bounded by rectification is curves.

First are some definitions:

A partition (or mesh) in an interval \([a, b]\), represented by \(R\), is a group \(P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}\) of interval points such that

\[
x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b
\]

The norm of a partition, denoted by \(\|P\|\), is the length of the largest subinterval of the partition.

It is called a regular partition of an interval \([a, b]\) where those subintervals all have the same length, referred to as step knitted \(h = (b - a) / n\).

The area of the region bounded by a function \(f(x)\) continuous and positive on an interval \([a, b]\), the \(x\)-axis (zero function) and the lines \(x = x = b\), as defined in the figure,

It is calculated by decomposition into rectangles:

1. Construir one partition \(P\) of \([a, b]\);

2. Encontrar in each subinterval \([x_{i-1}, x_i]\) the maximum \(M_i\) and minimum \(m_i\) whose existence is guaranteed by the Weierstrass theorem

\[
M_i = \max_{x \in [x_{i-1}, x_i]} f(x) \quad m_i = \min_{x \in [x_{i-1}, x_i]} f(x)
\]
3. Enquadrar The area between the upper and lower Riemann sums:

\[ \sum_{i=1}^{n} m_i \Delta x_i \leq A \leq \sum_{i=1}^{n} M_i \Delta x_i \]

where \( x_{i-1} \leq \xi_i \leq x_i \), ..., \( n \)

4. How the function is continuous, the limits of the upper and lower sums of Riemann are equal, then both are equal to the area that is to be calculated:

\[ \lim_{n \to \infty} \sum_{i=1}^{n} m_i \Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} M_i \Delta x_i = A \]

Example:

Calculate the area of the region bounded by the curve \( y = x^2 \), the x axis and the line \( x = 3 \).

With a regular partition of the interval \([0, 3]\) for \( n \) equal subintervals of length \( Ax = (3-0) / n \) = \( 3/n \), it follows that \( x_{i-1} = (i-1) \) and \( Ax = M_i ((i-1)^2) = (i-1)^2 (9/n^2) \)

So

\[ A = \lim_{n \to \infty} \sum_{i=1}^{n} m_i \Delta x_i \]

= \( \lim_{n \to \infty} \sum_{i=1}^{n} (i-1)^2 (9/(n^2)) (3/n) \)

= \( \lim_{n \to \infty} ((27)/(n^3)) \sum_{i=1}^{n} ((i-1)^2) \)

= \( \lim_{n \to \infty} ((27)/(n^3))[((n(n+1)(2n+1))/6)-2((n(n+1))/2)+n] \)

= 9

**Definite Integral and its Properties**

A function \( f(x) \) is (Riemann) integrable on an interval \([a, b]\) if there is a number \( L \) that satisfies the boundary condition:

\[ \forall \varepsilon > 0, \exists \delta > 0: \forall \Delta ||\Delta|| < \delta \implies \lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_i) \Delta x_i \leq L \leq \varepsilon , \text{ where } x_{i-1} \leq \xi_i \leq x_i \]

Under these conditions it is written

\[ \lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_i) \Delta x_i = L \]

If a function is continuous on an interval \([a, b]\) then it is (Riemann) integrable.

If a function has removable discontinuities (or jump) in only a countable number of points, then it is still Riemann integrable.
There are functions that are not (Riemann) integrable, i.e., where the upper limit and lower limit of Riemann are different. However, there is a more general definition of integration (Lebesgue) for these cases.

If a function is integrable on the closed interval \([a, b]\) then the definite integral of \(f\) from \(a\) to \(b\), is given by

\[
\int_{(a)}^{(b)} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_i) \Delta x_i \quad \text{onde} \quad x_{i-1} \leq \xi_i \leq x_i
\]

if the limit exists.

Ex: Calculate the full set of \(f(x) = x^3\), \(x = 1\) to \(x = 2\)

For a regular partition of \([1, 2]\) is taken \(\Delta x_i = (\frac{2-1}{n}) = \frac{1}{n}\)

For the \(i\)th \(f\) argument is taken \(\xi_i = x_i = 1 + i\Delta x_i\)

So

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_i) \Delta x_i = \lim_{n \to \infty} \left( \frac{1}{n} + \frac{(3i)}{n^2} + \frac{(3i^2)}{n^3} + \frac{(i^3)}{n^4} \right)
\]

\[
= 1 + \frac{3}{2} + 1 + \frac{1}{4}
\]

\[
= \frac{15}{4}
\]

The area of a region bounded by a function \(f(x)\) (Riemann) integrable and positive in an interval \([a, b]\), the x-axis (zero function) and the lines \(x = x = b\) is redefined as the definite integral of \(f(x)\) from \(a\) to \(b\)

The \(\int_{(a)}^{(b)} f(x) \, dx\)

List of some properties of definite integrals in the range \([a, b]\):

\[
\int_{(a)}^{(b)} f(x) \, dx = \int_{(a)}^{(b)} f(u) \, du = \int_{(a)}^{(b)} f(t) \, dt \quad \text{independent variable mute symbol integration ;}
\]

\[
\int_{(a)}^{(a)} f(x) \, dx = 0
\]

\[
\int_{(a)}^{(b)} f(x) \, dx = -\int_{(b)}^{(a)} f(x) \, dx
\]

\[
\int_{(a)}^{(b)} f(x) \, dx = \int_{(a)}^{(c)} f(x) \, dx + \int_{(c)}^{(b)} f(x) \, dx \quad \text{for} \, c \in [a, b]
\]

\[
\int_{(a)}^{(b)} KDX = k (ba) \quad \text{for constant} \, k
\]

\[
\int_{(a)}^{(b)} (af(x) + \beta g(x)) \, dx = \alpha \int_{(a)}^{(b)} f(x) \, dx + \beta \int_{(a)}^{(b)} \, g(x) \, dx \quad \text{linearity}
\]

If \(f(x) \geq g(x) \forall x \in [a, b]\) then the \(\int_{(b)}^{(a)} f(x) \geq \int_{(b)}^{(a)} g(x) \, dx
\]

If \(m \leq f(x) \leq M \), \(\forall x \in [a, b]\) then \(m \leq \frac{1}{(ba)} \int_{(a)}^{(b)} f(x) \, dx \leq M
\]
Theorem of average value for definite integrals

The last property listed the number \( \frac{1}{(b-a)} \int_{a}^{b} f(x) \, dx \) can be interpreted as the value metering \( f[a, b] \), and at equal height one of rectangle base \( (ba) \) and the same area in definite integral area:

\[
\left(\text{Average integrals}\right) \text{ If the function } f \text{ is continuous in the closed interval } [a, b], \text{ there is a number } \chi \in [a, b] \text{ such that}
\]
\[
\int_{a}^{b} f(x) \, dx = f(\chi)(ba)
\]

The statement verifies that if \( f(x) \) is a continuous function in a closed interval, then the Weierstrass theorem, \( f \) has maximum and minimum points in the interval \([a, b]\)

\[
m \leq f(x) \leq M, \quad \forall x \in [a, b]
\]

Last listed property,

\[
m \leq \frac{1}{(b-a)} \int_{a}^{b} f(x) \, dx \leq M
\]

By Theorem of Bolzano, \( f(x) \) reaches all values between the maximum and the minimum, then there is a \( \chi \in \) point \([a, b]\) such that

\[
f(\chi) = \frac{1}{(b-a)} \int_{a}^{b} f(x) \, dx
\]

The Fundamental Theorems of Calculus (TFC)

The area of a region whose value is given by the definite integral \( \int_{b}^{f(x)} \, dx \) depends only on the edges of this region, i.e. \( f(x) \) and the ends \( a \) and \( b \).

Consequently, if \( x \in [a, b] \) is the position of a movable board, then the mobile edge area

\[
F(x) = \int_{a}^{x} f(t) \, dt
\]

only defines the \( R \) portion of the area between and \( x \)
By convention the symbol \( (x) \) is used to represent the position of the movable board and the mute symbol of the definite integral becomes \( t \).

(First TFC) is \( f \) is a continuous function in the interval \([a, b]\) and cumulative function \( F(x) = \int_a^x f(t) \, dt \) where \( x \in [a, b] \).

So that

\[ F'(x) = f(x) \]

In the statement, as

\[ F(x + h) = \int_a^x f(t) \, dt + \int_x^{x+h} f(t) \, dt \]

Then the mean value theorem for definite integrals,

\[ F(x + h) - F(x) = \int_a^h f(t) \, dt \]

\[ = F(\chi) \text{ for any } \chi \in [x, x+h] \]

It follows that

\[ F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \to 0} \frac{f(\chi)h}{h} = f(x) \]

Example:

One has to \( \frac{d}{dx} \left( \int_a^x \ln t \, dt \right) = \ln x \)

But to calculate \( \frac{d}{dx} \left( \int_a^e^{x} \ln t \, dt \right) \) we must first make a change of variable \( u = e^x \) \[ \Rightarrow \left( \frac{du}{dx} / (du) \right) = e^x \]

\( \frac{d}{dx} \left( \int_a^e^{x} \ln t \, dt \right) = \left( \frac{d}{du} \left( \int_a^u \ln t \, dt \right) \right) \cdot \frac{du}{dx} \)

\[ = (\ln e^x)e^x \]

\[ = xe^x \]
The second TFC establishes the connection between the integral defined as the primitive of an exact differential form.

(Second TFC or Barrow rule) If \( f(x)dx \) is an exact form in the range \([a, b]\), that is, if a function \( f(x) \) such that \( dF(x) = f(x)dx \) So

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

In the demonstration, let \( F(x) \) is such that \( dF(x) = f(x) \, dx \)

By definition of exterior derivative one has to

\[
(D/\, dx) \, F(x) = f(x)
\]

and the first TFC

\[
(\, D/\, dx) \, (\int_{a}^{x} f(t) \, dt) = f(x)
\]

Then the difference between the two can be a constant

\[
F(x) = \int_{a}^{x} f(t) \, dt + c
\]

As

\[
F(a) = \left\{A \int_{a}^{a} f(t) \, dt + c = c
\]

and

\[
F(b) = \int_{a}^{b} f(t) \, dt + c
\]

So

\[
F(b) - F(a) = \int_{a}^{b} f(t) \, dt
\]

The second TFC is used to calculate areas only using the primitive, without resorting to a limit of Riemann sums!

Example:

Calculate the area of the region bounded by the curve \( y = x^2 \), the x axis and the line \( x = 3 \).

Since the primitive is given by \( \int x^2 dx = \left( x^3 / 3 \right) + c \), the area is calculated

\[
\int_{0}^{3} x^2 dx = \frac{(3^3)}{3} = 9
\]

Example:

Calculate the definite integral of \( f(x) = x^3 \) \( x = 1 \) to \( x = 2 \)

How primitive is given by \( \int x^3 dx = \left( x^4 / 4 \right) + c \), the definite integral is calculated

\[
\int_{1}^{2} x^2 dx = \frac{(2^4)}{4} - \frac{(1^4)}{4} = 4 - (1/4) = (15)/4
\]

full set of applications

Calculation of area of plane figures

The Riemann sum calculates a flagged area.
If the function of the signal varies in the range \([a, b]\), the sign of the area varies accordingly.

Example:

The area given by \(\int_0^{2\pi} \sin x = -\cos 2\pi - (-\cos 0) = 0\) for while \(\int_0^{\pi} \sin x = 1\) the other half \(\int_{\pi}^{2\pi} \sin x = -1\)

The area of a region of the \(xy\) plane bounded by curves are graphs of functions \(f(x)\) \(g(x)\) on an interval \([a, b]\) is given by

\[\int_a^b (f(x) - g(x)) \, dx\]

It is positive in regions where \(f(x) > g(x)\) and negative otherwise.

The full set \(\int (a) \{ b \} f(x) \, dx\) is a special case where \(g(x) = 0, \forall x \in [a, b]\) is the null function.

Example:

The area bounded by the curves \(y = x^2\) \(y = 4x + -x^2\) is given by

\[
\int_0^2 (-x^2 + 4x - x^2) \, dx = \int_0^2 (-2x^2 + 4x) \, dx
\]

\[
= [- (2/3) x^3 + 2x^2] = x^2 \{ 2 \} - [- (2/3) x^3 + 2x^2] \{ x \} = 0
\]

\[
= - ((16 / 3) 8 = (8/3)
\]
The limits of integration correspond to the abscissa of the points of intersection of the two curves.

In the area of calculating a portion of the xy plane bounded by simple curves given by the variable equations (x, y), the curves can match function graphs (implicit) only vertically simple regions (vertical bands) where it is possible to calculate \(( f(x) - g(x)) \, dx\)

or horizontally simple regions (horizontal stripes) where you can calculate \((h(y) - k(y)) \, dy\)

Example:

The area of the region bounded by \(2x - y^2 - 2\) equations \(= 0\) and \(xy - 5 = 0\)

The intersection of the two curves is obtained by substituting \(y = 5\), \(x\) in the first equation and solving

\[2x - (x - 5)^2 - 2 = 0\]

Obtain \(x = 3\) \(x = 9\)

The equation \(2x - y^2 - 2 = 0\) implicitly defines two functions: \(y = \sqrt{2x - 2}\) \(y = -\sqrt{2x - 2}\) both in the field \([1, +\infty)\)

The area calculated in the two regions vertically simple \(x \in [1, 3]\) \(x \in [3, 9]\) results in

\[
\int_1^3 (\sqrt{2x - 2} - (-\sqrt{2x - 2})) \, dx + \int_3^9 (\sqrt{2x - 2} - (5 - x)) \, dx
\]

\[
= \left( \frac{2}{3} (2x - 2)^{3/2} \right)_1^3 + \left( \frac{1}{3} (2x - 2)^{3/2} \right)_3^9 - \left( \frac{(x^2)}{2} - 5x \right)_3^9
\]

\= 18
The area calculated in only horizontally simple \( y \in \) region \([-2, 4]\) results in

\[
\int_{-2}^{4} \left( (y + 5) - \left( \frac{y^2}{2} + 1 \right) \right) \, dy = \left[ \frac{y^2}{2} + 4y - \frac{y^3}{6} \right]_{-2}^{4} = 18
\]

Volume solids by cutting, circular disk and cylindrical housings

Solid cylindrical are obtained by inside the extrusion of a region \( R \) bounded by a curve from one plane to another parallel plane.

If extrusion is perpendicular to the parallel planes, the cylindrical solid tell its straight.

![Diagram of a cylindrical solid]

If the area of the region \( R \) is invariant with extrusion, the solid volume \( V \) is the base area multiplied by the height \( h \) of the solid (extrusion length).

\[
V = Ah
\]

If at each step of extruding the curve deform in a continuous manner, area \( A(x) \) of a solid cutting it depends on the position \( x \) and extrusion volume

the solid is given by

\[
V = \int_{0}^{h} A(x) \, dx
\]

Even though extrusion is not perpendicular to the solid volume is given by the same formula, \( h \) is measured perpendicularly to the parallel planes as the Cavalieri’s principle, if the sectional area is equal in all stages, the volume of the two solids is also the same.

Example:

Use a cut to find the volume of a straight pyramid whose height is \( h \) and has square base side \( s \).

The proportion of triangles, cutting a \( y \) position of the base is a square side equal to \((s/h)(HX)\), so the sectional area of that position is \( A(y) = \left( \frac{s^2}{h^2} \right) (hX)^2 \).

The volume of right quadrangular pyramid is given by

\[
V = \int_{0}^{h} \left( \frac{s^2}{h^2} \right) (hy)^2 \, dy
\]

\[
= \left[ -\left( \frac{s^2}{h^2} \right) \left( \frac{(hy)^3}{3} \right) \right]_{y=0}^{y=h}
\]

\[
= \frac{1}{3} s^2 h
\]
A solid of revolution is obtained by rotating a plane in a region around an axis of revolution that may or may not intersect that region.

Taking area elements perpendicular to the axis of revolution, the volume element is a circular disc whose variable radius is given by the function \( f(x) \), and the volume can be calculated by

\[
\pi \int_{a}^{b} [f(x)]^2 \, dx
\]

Example:

The volume of the solid obtained by the rotation about the \( x \) axis, in a region under the curve \( y = \sqrt{x} \) from \( 0 \) to \( 1 \), is given by

\[
V = \pi \int_{0}^{1} [\sqrt{x}]^2 \, dx
\]

\[
= \pi \int_{0}^{1} x \, dx
\]

\[
= \pi \left[ \frac{x^2}{2} \right]_{0}^{1}
\]

\[
= \frac{\pi}{2}
\]

If the area element perpendicular to the axis of revolution does not intersect with the axis of revolution, the volume element is an annulus with inner radii and outer variable data by the functions \( f(x) \) \( g(x) \), the volume can be calculated per

\[
V = \pi \int_{a}^{b} (f(x)^2 - g(x)^2) \, dx
\]
Example:

The volume of the R region bounded by the curves $y = x$ and $y = x^2$ and rotated around the x-axis is given by

$$V = \pi \int_{a}^{b} ((x)^2 - (x^2)^2) \, dx$$

$$= \pi \left[ \left( \frac{x^3}{3} \right) - \left( \frac{x^5}{5} \right) \right]_{a}^{b} = \left( \frac{2\pi}{15} \right)$$

Taking area elements parallel to the axis of revolution obtains a cylindrical casing at a distance $x$ from the axis of revolution with time variable $f(x)$ and whose volume is given by

$$V = 2\pi \int_{a}^{b} xf(x) \, dx$$

Example:

The same volume of a solid obtained by rotating about the x-axis, in a region under the curve $y = \sqrt{x}$, from 0 to 1, now calculated for cylindrical shells results in

$$V = 2\pi \int_{0}^{1} y(1-y^2) \, dy$$

$$= 2\pi \left[ \left( \frac{y^2}{2} \right) - \left( \frac{y^4}{4} \right) \right]_{0}^{1}$$

$$= \left( \frac{\pi}{2} \right)$$

**Arc length**

The length of arc of a function $f(x)$ in the interval $[a, b]$, is calculated by the same infinitesimal approximation method already used to calculate derivatives and areas. With a partition interval $[a, b]$ with $||\Delta||$ standard, they add up the length of segments.
In the limit when the \( \|\Delta\| \) to zero, the arc length is given, if the limit exists for

\[
L = \lim_{\|\Delta\| \to 0} \sum_{i=0}^{n} |P_i - P_{i-1}|
\]

As \( |P_i - P_{i-1}| \) it can be estimated with the aid of the mean value theorem applied to \( f(x) \) in the interval \([x_i, x_{i-1}]\)

for

\[
|P_i - P_{i-1}| = \sqrt{(\Delta_i x)^2 + (\Delta_i y)^2} = \sqrt{(\Delta_i x)^2 + (f'(x_i) \Delta_i x)^2}
\]

\(x_i \in [x_i, x_{i-1}]\)

The arc length is given by the Riemann integral

\[
L = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx
\]

Example:

The arc length of the curve \( y = x^{(2/3)} \) of the point \((1,1)\) to \((8,4)\) is given by

\[
L = \int_{1}^{8} \sqrt{1 + (2/3 x^{(-1/3)})^2} \, dx
\]

\[
= (1/3) \int_{1}^{8} (\sqrt{(9x + (2/3)) + 4} / (x^{(2/3)}) \, dx
\]

By the substitution method

\[
9x = u^{(2/3)} + 4
\]

\[
du = 6x^{(2/3)} \, dx
\]
is obtained
\[ L = \frac{1}{18} \int_{13}^{40} u^{(1/2)} \, du \]
\[ = \frac{1}{27} [40^{(3/2)} - 13^{(3/2)}] \]
\[ \approx 7.6 \]

Mass Moment and Centroid of Flat Land

Consider a homogeneous sheet of uniform density \( k \) across a planar region bounded by the curve \( y = f(x) \), the \( x \)-axis by the straight lines \( x = a \) and \( x = b \).

The total mass of the blade is given by
\[ K \int_a^b f(x) \, dx \]

The mass moment of \( L \) depends on the chosen blade shaft in relation to the rotation axis \( y \) is given by
\[ m_y = k \int_a^b xf(x) \, dx \]

The axis \( x \) of rotation is given by
\[ m_x = \frac{k}{2} \int_a^b [f(x)]^2 \, dx \]

The center of mass of the blade is independent of the density and is therefore designated centroid is given by
\[ x = \frac{\left( \int_a^b xf(x) \, dx \right)}{m} \]
\[ y = \frac{\left( \int_a^b [f(x)]^2 \, dx \right)}{2m} \]

Conclusion

This activity dealt with the Riemann integral and its properties, particularly important theorems presented the mean value theorem for definite integrals and fundamental theorems of Calculus. the definite integral applications were studied: figures Area Calculation planar, bulk solids by cutting circular disk and cylindrical shells, arc length, mass centroid moment and flat region.
Evaluation

Group a minimum of 5 and a maximum of 10 questions in each of the following 1 hour evaluation test without reference.

Choose different type questions in the preparation of each test, the changing parameters and database to prevent bonding.

Each question is worth between 10% and 20% depending on the number of selected questions.

Question 1: Use the method of infinitesimal approach to calculate the area of the region bounded by \( y = x^3 + 3 \), x-axis, and line \( x = 0 \) and \( x = 2 \);

Answer:

\[
\Delta_x^i = \frac{(2 - 0)}{n} = \frac{2}{n}
\]

\[
\Delta_x^i = X_i - x_{i-1}
\]

\[
x_i = i\Delta_x = \frac{(2i)}{n} \text{ for } i = 1, ..., n
\]

One can use both \( f(x_i) \) and \( F(x_{i-1}) \) to the height of the rectangles, and \( \Delta_x^i \) for the base of the rectangles:

The \( \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta_x^i \)

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \left( x_i^3 + 3 \right) \Delta_x^i
\]

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{16i^3}{n^4} \right) + \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{6}{n} \right)
\]

\[4 + 6 = 10\]

Question 2: Find the exact area of the region bounded by \( y = x^3 \), \( y = 0 \), \( x = 1 \) and \( x = 2 \)

\[
\int_{1}^{2} x^3 dx = \left[ \frac{x^4}{4} \right]_1^2
\]

\[= \left( \frac{2^4}{4} \right) - \left( \frac{1^4}{4} \right)\]

\[= 4 - \left( \frac{1}{4} \right)\]

\[= \left( \frac{15}{4} \right)\]

Question 3: Find the exact area of the region bounded by \( y = \frac{1}{x} \), \( y = 0 \), \( x = 3 \) and \( x = 4 \)

Answer:

\[
\int_{3}^{4} \left( \frac{1}{x} \right) dx = \ln x \bigg|_{3}^{4}
\]

\[= \ln 4 - \ln 3\]

\[= \ln \left( \frac{4}{3} \right) = 0.28768\]
Question 4: Find the exact area of the region bounded by $y = \frac{1}{x}$, $y = 0$, $x = 3$ and $x = 4$

Answer:

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x \, dx$$

How to find $\int \tan x \, dx$?

Note that the exterior derivative

$$DSEC \ x = sec \ x \ dx \ x tan$$

Soon

$$\tan \ x \ dx = \left( \frac{DSEC \ x}{sec \ x} \right) = d \ln | sec \ x |$$

So

$$\int \tan \ x \ dx = \int d \ln | sec \ x |$$

$$= \ln | sec \ x | + c$$

Hence they obtain the requested area

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x \, dx = \ln | sec \ \left( \frac{\pi}{3} \right) | - \ln | sec \ \left( -\frac{\pi}{6} \right) |$$

$$= \ln 2 - \ln \sqrt{3}$$

$$= 0.54931$$

Question 5: Use the mean value theorem for integrals to prove $\int_0^2 \frac{1}{(x^2 + 4)} \, dx \leq \frac{1}{2}$

Answer: From the range $[0, 2]$, the function lies between $0 \leq \frac{1}{(x^2 + 4)} \leq \frac{1}{4}$, then the average value of the function must be between the same values:

$$0 \leq \left( \frac{\int_0^2 \left( \frac{1}{(x^2 + 4)} \right) \, dx}{2-0} \right) \leq \frac{1}{4} \iff 0 \leq \int_0^2 \left( \frac{1}{(x^2 + 4)} \right) \, dx \leq \frac{1}{2}$$

Question 6: Find the average value of the function $f(x) = x$ given in the range $[-1, 2]$, and give an $\chi$ point where $f(\chi)$ reaches that average.

Answer:

$$\int_{-1}^{2} x \, dx = \left( \frac{x^2}{2} \right)_{-1}^{2}$$

$$= \left( \frac{2^2}{2} \right) - \left( \frac{(-1)^2}{2} \right)$$

$$= \frac{3}{2}$$

$$f(\chi) = \left( \frac{(3/2)}{2 - (-1)} \right)$$

$$= \frac{1}{2} \quad \chi = \frac{1}{2}$$
Question 7: Calculate the derivatives of the mobile board integrals \( \frac{d}{dx} \left( \int_0^x \sqrt{4 + t^6} \, dt \right) \)

Answer: \( \sqrt{x^6 + 4} \)

Question 8: Calculate the derivatives of the mobile board integrals \( \frac{d}{dx} \left( \int_x^3 \sqrt{1 + t^4} \, dt \right) \)

Answer: \(-\sqrt{x^4 + 1}\)

e) \( \frac{d}{dx} \left( \int_{-x}^x x^2 \cos(t^2 + 1) \, dt \right) \)

Question 9: Derive the formula for the volume of a right circular cone with height \( h \) units and the base units of rays, using a cut.

A. Placing the cone apex on the origin and the cone axis in the x-axis, the cone radius is given by \( f(x) = \left(\frac{a}{h}\right)x \) and the sectional area is \( \pi \left(\frac{a}{h}\right)^2 x \), logo

\[
V = \int_0^h \pi \left(\frac{a}{h}\right)^2 x \, dx
= \left[ \pi \left(\frac{a^2}{h^2}\right) \frac{x^3}{3} \right]_0^h
= \left(\frac{\pi a^2}{3} \right) \frac{h^3}{h^2}
= \frac{\pi a^2 h}{3}
\]

Question 10: Compute the sound volume generated by the rotation of the region bounded by the arc sine curve around the x axis.

Answer:

\[
\int_0^\pi \pi \sin^2 x \, dx = \left(\frac{\pi}{2}\right) \int_0^\pi (1 - \cos 2x) \, dx
= \left(\frac{\pi}{2}\right) \left( x - \frac{\sin 2x}{2} \right)_0^\pi
= \left(\frac{\pi^2}{2}\right)
\]

Note: the three basic formulas support integral calculus of trigonometric functions:

\[
\cos \{ x \} = 1 + \sin^2 x
\]
\[
\cos (a + b) = \cos a \sin b + \sin a \cos b - b
\]
\[
\sin (a + b) = \sin a \sin b + \cos a \cos b - b
\]

From the formula of the sum, when \( a = b \) is used upon the replacement

\[
\sin^2 a = \left(\frac{1}{2}\right) (1 - \cos 2a)
\]
Summary

This activity showed up the indispensable tools in the application of the calculation method: the primitives of elementary functions, the integral of a differential form, the fundamental theorem of calculus, integration techniques, the Riemann integral, the geometric interpretation of the integral Riemann and calculation of areas and volumes using elementary integration.

Also presented some applications such as the calculation area of plane figures, volume solids by cutting, circular disk and cylindrical shells, arc length, mass centroid moment and flat region.

Activity Rating

Instructions

Group a minimum of 5 and a maximum of 10 questions in each of the following evaluation test 2 hours without appointments.

Choose different type of questions in the preparation of each test, the changing parameters and database to prevent bonding.

Activity 1

Question 1: Do the following primitivações : \( \int (3-2t + t^2) \, dt \)

Answer:

\[
\int (3-2t + t^2) \, dt = 3t - t^2 + \left( \frac{t^3}{3} \right) + c
\]

Verification: \( \left( \frac{d}{dt} \right) (3t - t^2 + \left( \frac{t^3}{3} \right) + c) = t^2 - 2t + 3 \)

Question 2: Do the following primitivações : \( \int \left[ \frac{3}{3} \sqrt{x} + \left( \frac{1}{\frac{3}{3} \sqrt{x}} \right) \right] dx \)

Answer:

\[
\int \left[ \frac{3}{3} \sqrt{x} + \left( \frac{1}{\frac{3}{3} \sqrt{x}} \right) \right] dx = \int \left( x^{\frac{1}{3}} + x^{\frac{1}{3}} \right) dx
\]

\[
= \left( \frac{3}{4} \right) x^{\frac{2}{3}} + \left( \frac{3}{2} \right) x^{\frac{2}{3}} + c
\]

Verification: \( \left( \frac{d}{dx} \right) \left[ \left( \frac{3}{4} \right) x^{\frac{2}{3}} + \left( \frac{3}{2} \right) x^{\frac{2}{3}} + c \right] = \left( \frac{3}{4} \right) \sqrt{x} + \left( \frac{3}{2} \right) \sqrt{x} + c \)

Question 3: Do the following primitivações : \( \int x^3 (2- x^2)^{12} \, dx \)

Answer: \( \int x^3 (2- x^2)^{12} \, dx \)

Let \( u = 2- x^2 \Rightarrow du = -2x \, dx \)

\[
\int x^3 (2- x^2)^{12} \, dx = -\frac{1}{2} \int x^2 (2- x^2)^{12} \, dx
\]

\[
= -\frac{1}{2} \int (u - 2)^{12} \, du
\]

\[
= -\left( \frac{1}{13} \right) (2u^{12} - u^{13}) \, du
\]
\[
\begin{align*}
= & - (\frac{1}{2}) \left(\frac{2u^{13}}{13}\right) + (\frac{1}{2}) \left(\frac{u^{14}}{14}\right) + c \\
= & \left(\frac{2 - x^2}{13}\right) + \left(\frac{2 - x^2}{28}\right) - (\frac{1}{13}) + c \\
= & (2 - x^2)^{13} \left(\frac{(2 - x^2)}{28}\right) - (\frac{1}{13}) + c
\end{align*}
\]

Verification: \( \frac{d}{dx} \left(\frac{(2 - x^2)^{13}((2 - x^2))}{28} - (\frac{1}{13}) + c\right) = X^3 (x^2 - 2)^2 \)

Question 4: Do The Following primitives: \( \int \left(\frac{x^3}{\sqrt{1 - 2x^2}}\right) dx \)

Answer: \( \int \left(\frac{x^3}{\sqrt{1 - 2x^2}}\right) dx \)

Let \( u = 1 - 2x^2 \)
\(\int \left(\frac{x^3}{\sqrt{1 - 2x^2}}\right) = - (\frac{1}{4}) \int \left(\left(\frac{-4x}{\sqrt{1 - 2x^2}}\right)\right) \\)
\(= - (\frac{1}{4}) \int \left(\left(\frac{u}{2}\right) du\right) \left(\sqrt{u}\right) \)
\(= - (\frac{1}{8}) \int \left(\sqrt{u}\right) \left(-\frac{1}{2}\right) u^2 \\left(\frac{1}{\sqrt{u}}\right) du \)
\(= - (\frac{1}{8}) \int \left(\sqrt{u}\right) \left(-\frac{1}{2}\right) u^2 \left(\frac{1}{\sqrt{u}}\right) + c \)
\(= - (\frac{1}{8}) \int \left(\sqrt{u}\right) \left(-\frac{1}{2}\right) u^2 \left(\frac{1}{\sqrt{u}}\right) + c \)

Verification: \( \frac{d}{dx} \left(- (\frac{\sqrt{u}}{4}) + (\frac{1}{12}) \sqrt{u^3} + c\right) = \left(\frac{x^3}{\sqrt{1 - 2x^2}}\right) \)

Question 5: Do the following primitives: \( \int \left(\frac{x^3}{(1 - 2x^4)^5}\right) dx \)

Answer: \( \int \left(\frac{x^3}{(1 - 2x^4)^5}\right) dx \)

Let \( u = 1 - 2x^4 \)
\(\int \left(\frac{x^3}{(1 - 2x^4)^5}\right) = - (\frac{1}{8}) \int \left(\left(\frac{-8x^3}{(1 - 2x^4)^5}\right)\right) \\)
\(= - (\frac{1}{8}) \int \left(\left(\frac{u}{2}\right) du\right) \left(\frac{u^5}{2}\right) \)
\(= - (\frac{1}{8}) \int u^{-5}du \)
\(= - (\frac{1}{8}) \left(\frac{u^{-4}}{-4}\right) + c \)
\(= (\frac{1}{32}) \left(\frac{x^3}{(1 - 2x^4)^4}\right) + c \)

Verification: \( \frac{d}{dx} \left((\frac{x^3}{(1 - 2x^4)^5}) + c\right) = \left(\frac{x^3}{(1 - 2x^4)^5}\right) \)

Question 6: Do the following primitivações: \( \int (\tan 2x + \cot 2x)^2 dx \)

Answer:
\(\int (\tan 2x + \cot 2x)^2 dx = \int (\tan 2x + 2tan 2xcot 2x + \cot 2x + 2cot 2x + 2cot 2x) dx \)
\(= (\frac{1}{2}) \tan 2x - (\frac{1}{2}) \cot 2x + 2\tan 2xcot 2x + c \)
\(= (\frac{1}{2}) \tan 2x + (\frac{1}{2}) + c \cot 2x \)
For each of the primitive results in :
\[ \int \tan^2 2x \, dx = \int (\sec^2 2x - 1) \, dx \]
\[ = \frac{1}{2} x + c - \tan 2x \]
\[ \int \cot^2 2x \, dx = \int (\csc^2 2x - 1) \, dx \]
\[ = - \frac{1}{2} x + c - \cot 2x \]
Verification: \( \frac{d}{dx} \left( \frac{1}{2} \tan 2x - \frac{1}{2} \cot 2x + c \right) = \cot^2 2x \tan^2 2x + 2 \)

Question 7: Do the following primitivações : \( \int \left( \frac{(\sqrt{x} - 1)^2}{\sqrt{x}} \right) \, dx \)
Answer : \( \int \left( \frac{(\sqrt{x} - 1)^2}{\sqrt{x}} \right) \, dx = \int \left( \frac{x - 2\sqrt{x} + 1}{\sqrt{x}} \right) \, dx \)
\[ = \int \left( x^{\frac{1}{2}} - 2 + x^{-\frac{1}{2}} \right) \, dx \]
\[ = \left( \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) - 2x + \left( \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right) + c \]
\[ = \frac{2}{3} x \sqrt{x} - 2x + c + 2\sqrt{x} \]
Verification: \( \frac{d}{dx} \left( \frac{2}{3} 2x + x\sqrt{x} - 2\sqrt{x} + c \right) = \left( \frac{1}{\sqrt{x}} \right) \left( \sqrt{x} - 1 \right)^2 \)

Question 8: Find the solution of the differential equation :
\( ( \frac{d}{dx} y ) / (dx) = 2x + 3x^2 - 7 \)
Answer:
\( ( \frac{d}{dx} y ) / (dx) = 2x + 3x^2 - 7 \)
\( dy = ( ( dy ) / (dx)) \, dx \)
\[ = (2x + 3x^2 - 7) \, dx \]
\[ = D \left( x^3 + x^2 - 7x \right) \]
Soon
\( y = x^2 - x^3 + 7x + c \)
Verification: \( ( d / (dx) ) ( x^3 + x^2 - 7x + c ) = 2x + 3x^2 - 7 \)

Question 9: Find the solution of the differential equation :
\( ( \frac{d^2}{dx^2} y ) / (dx^2) + 1 = 5x^2 \)
Answer. \( ( \frac{d^3}{dx^3} y ) / (dx^3) = 5x^2 \) \( \iff \) \( ( d / (dx) ) (( dy ) / (dx)) + 1 = 5x^2 \)
Leave
\( ( dy ) / (dx) = u \)
Soon
\( ( Du ) / (dx) + 1 = 5x^2 \)
Solving the first equation of 1st order

\[ du = (\frac{du}{dx}) \, dx \]

\[ = (5x^2 + 1) \, dx \]

\[ D = (\frac{5}{3} x^3 + x) \]

Soon

\[ u = (\frac{5}{3} x^3 + x + c_1) \]

replacing \( u \)

\[ (\frac{Dy}{dx}) = (\frac{5}{3} x^3 + x + c_1) \]

Solving the second equation of 1st order

\[ dy = (\frac{dy}{dx}) \, dx \]

\[ = (\frac{5}{3} + x + x^3 c_1) \, dx \]

\[ D = ((5 / (12)) x^4 + ((x^2) / 2) + c_1 x) \]

Soon

\[ y = (5/(12))x^4 + ((x^2)/2) + c_1 x + c_2 \]

10 - Question: Find the solution of the differential equation determined by the initial conditions:

\[ \frac{(D^2 u)}{(dv^2)} = 4 (1 + 3v)^2; \text{ initial conditions: } u (-1) = -1 \text{ and } \frac{(du)}{(dv)} (-1) = -2 \]

Answer:

\[ \frac{(D^2 u)}{(dv^2)} = 4 (1 + 3v)^2 \Rightarrow (d / (dv)) (((du) / (dv))) = 4 (1 + 3v)^2 \]

Leave

\[ (Du) / (dv) = y \]

Soon

\[ (Dy) / (dv) = 4 (1 + 3v)^2 \]

Solving the first equation of 1st order

\[ dy = (((dy) / (dv))) \, dv \]

\[ 4 = (1 + 3v)^2 \, dv \]

\[ D = ((4/9) (1 + 3v)^3) \]

Soon

\[ y = (4/9) (1 + 3v)^3 + c_1 \]

Replacing the first initial condition \( (du) / (dv)) (-1) = -2 \Rightarrow y (-1) = -2 \]

\[ -2 = (4/9) (1 + 3 (-1))^3 + c_1 \Leftrightarrow c_1 = ((14) / 9) \]
replacing y

\[(\frac{Du}{dv}) = \frac{4}{9} (1 + 3v)^3 + \frac{14}{9}\]

Solving the second equation of 1st order

\[du = \left(\frac{du}{dv}\right) dv\]

\[= \frac{4}{9} (1 + 3v)^3 + \frac{14}{9} dv\]

\[D = \frac{1}{(27)} (1 + 3v)^4 + \frac{14}{9} v\]

Soon

\[u = \frac{1}{(27)} (1 + 3v)^4 + \frac{14}{9} v + c_2\]

Replacing the second initial condition \(u(-1) = -1\)

\[-1 = \frac{1}{(27)} (1 + 3(-1))^4 + \frac{14}{9} (-1) + c_2\]

Soon

\[u = \frac{1}{(27)} (1 + 3v)^4 + \frac{14}{9} v (1 / 27)\]

**Activity 2**

Question: Find the five following elements of partial sums \(s_n\) and get a formula for the general term \(s_n\). Determine also the infinite series is convergent or divergent; if convergent determine their sum: \(\Sigma \{ n = 1 \} ^{\infty}\)

Answer:

\[\Sigma \{ n = 1 \} ^5 n = 1 + 2 + 3 + 4 + 5 = 15\]

\[s_n = \Sigma \{ k = 1 \} ^n k\]

\[= \frac{(N(n + 1))}{2}\]

Calculating the limit

\[\lim \{ n \rightarrow \infty \} \frac{(n(n + 1))}{2} = + \infty \cdot \text{divergent}\]

Question 2: Find the five following elements of partial sums \(s_n\) and get a formula for the general term \(s_n\). Determine also the infinite series is convergent or divergent; Convergent determine if the sum: \(\Sigma \{ n = 1 \} ^{\infty}\) \(\log(n/n+1)\)

Answer:

\[\Sigma \{ n = 1 \} ^5 \ln(n/n+1) = \ln(1/2) + \ln(2/3) + \ln(3/4) + \ln(4/5) + \ln(5/6)\]

\[s_m = \{ \Sigma \{ n = 1 \} ^m \} \ln(n/n+1)\]

\[\ln^{\Pi} \{ n = 1 \} ^m \{ n / (n + 1) \}\]

\[= \ln(1/(m + 1))\]

Calculating the limit

\[\lim \{ m \rightarrow \infty \} \ln(1/(m + 1)) = - \infty \cdot \text{divergent}\]
Question 3: Determine whether the series is convergent or divergent, if convergent find their sum: $\sum_{n = 1}^{\infty} \left(\frac{4}{3}\right) \left(\frac{5}{7}\right)^n$

Answer:

$$\sum_{n = 1}^{\infty} \left(\frac{4}{3}\right) \left(\frac{5}{7}\right)^n = \left(\frac{4}{3}\right) \sum_{n = 1}^{\infty} \left(\frac{5}{7}\right)^n$$

$$= \left(\frac{4}{3}\right) \left(\frac{\left(\frac{5}{7}\right)}{1 - \left(\frac{5}{7}\right)}\right)$$

$$= \left(\frac{10}{3}\right)$$

Question 4: Determine whether the series is convergent or divergent, convergent if the sum found:

$$\sum_{1}^{\infty} \left(\frac{2}{n(n + 2)}\right)$$

Answer:

$$\sum_{1}^{\infty} \left(\frac{2}{n(n + 2)}\right) = \sum_{n = 1}^{\infty} \left(\frac{1}{n} - \frac{1}{n + 2}\right)$$

$$= 1 + \left(\frac{1}{2}\right) - 2\lim_{n \to \infty} \left(\frac{1}{n}\right)$$

$$= \left(\frac{3}{2}\right)$$

Question 4: Determine whether the series is convergent or divergent, if convergent find their sum: $\sum_{n = 1}^{\infty} \left(2^{-n} + 3^n\right)$

Answer:

$$\sum_{n = 1}^{\infty} \left(2^{-n} + 3^n\right) = \sum_{n = 1}^{\infty} 2^{-n} + \sum_{n = 1}^{\infty} 3^n$$

soon diverges because the nature of a series is not changed if coupled with a convergent series

Question 5: Determine if the given Beheerder series is convergent or divergent:

$$\sum_{n = 1}^{\infty} \left(\frac{n!}{(2n)!}\right)$$

Answer:

$$0 \leq \left(\frac{n!}{(2n)!}\right) = \left(\frac{1}{(n + 1) \cdots (2n)}\right) \leq \left(\frac{1}{2^n}\right)$$

$$0 \leq \sum_{n = 1}^{\infty} \left(\frac{n!}{(2n)!}\right) \leq \sum_{n = 1}^{\infty} \left(\frac{1}{2^n}\right) = 1$$

Soon

$$\sum_{n = 1}^{\infty} \left(\frac{n!}{(2n)!}\right) Converges$$

For the sequence of Parcias sums (positively) is increasing and bounded.
Question 6: Determine if the given series is convergent or divergent: \( \sum_{n = 1}^{\infty} \frac{(2^n)}{(n)!} \)

Answer:

The ratio test

\[ \lim_{n \to \infty} \left| \frac{\frac{(2^{n+1})}{(n+1)!}}{\frac{(2^n)}{(n)!}} \right| = \lim_{n \to \infty} \frac{2}{n+1} = 0 \]

Question 7: Determine whether the alternating series is convergent or divergent: \( \sum_{n = 1}^{\infty} (-1)^{n+1} \frac{(n^2)}{(n^3 + 2)} \)

Answer:

\( \sum_{n = 1}^{\infty} (-1)^{n+1} \frac{(n^2)}{(n^3 + 2)} \) converges as the general term is decreasing and infinitesimal

For \( n > 2 \) has-

\[ 4n + 2 + \leq n^4 n^2 \iff \]

\[ N^4 \leq 4n + 2 + 2n^3 n^2 \iff \]

\[ n^5 2n^4 + + n^3 2n^2 + 4n + 2 \leq n^5 3n^4 + + 3n^3 3n^2 \iff \]

\[ (N + 1)^2 (n^3 + 2) \leq (n + 1)^3 + 2 \iff \]

\[ (((N + 1)^2) / ((n + 1)^3 + 2)) \leq ((N^2) / (n^3 + 2)) \]

\[ \lim_{n \to \infty} (N^2) / (n^3 + 2) = 0 \]

Question 8: Determine whether the alternating series is convergent or divergent: \( \sum_{n = 1}^{\infty} (-1)^n \frac{(e^n)}{n} \)

Answer:

\( \sum_{n = 1}^{\infty} (-1)^n \frac{(e^n)}{n} \) diverges as the general term is not decreasing even an infinitesimal

\[ \frac{((e^{n+1}) / (n + 1))) / (((e^n) / n))) = ((e^{n+1}) / (n + 1) e^n)) \]

\[ = ((Ne) / (n + 1))> 1 \]

\[ \lim_{n \to \infty} (-1)^n \frac{(e^n)}{n} \neq 0 \]

Question 9: Determine whether the alternating series is convergent or divergent: \( \sum_{n = 1}^{\infty} (-1)^n \frac{(\ln n)}{n} \)

Answer:

\( \sum_{n = 1}^{\infty} (-1)^n \frac{(\ln n)}{n} \) converges as the general term is decreasing and infinitesimal

\[ \frac{((\ln (n + 1))) / ((n + 1))) / (((\ln n) / n))) = ((\ln (n + 1)) / ((n + 1) \ln n)) \]

\[ = (\frac{\ln (n + 1)}{n}) / (\ln n^{(n+1)}) \leq 1 \]
lim_{n \to \infty} \left( \frac{(\ln n)}{n} \right) = 0

Question 10: Determine if the given series is absolutely convergent, conditionally convergent or divergent: \( \sum_{n = 1}^{\infty} (-\left(\frac{2}{3}\right)^n) \)

Answer:

\( \sum_{n = 1}^{\infty} (-\left(\frac{2}{3}\right)^n) \) is absolutely convergent, because

\[ \sum_{n = 1}^{\infty} \left(\left|\frac{2}{3}\right|^n\right) = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 2 \]

Pergunta 11: Determine if a series is absolutely convergent, convergent or divergent condicionalmente: \( \sum_{n = 1}^{\infty} \left(\frac{(1 + \frac{1}{n})^2}{e^n}\right) \)

Resposta: \( \sum_{n = 1}^{\infty} \left(\frac{(1 + \frac{1}{n})^2}{e^n}\right) \) convergent e pelo absolutamente tests da raiz

\[ \lim_{n \to \infty} \sqrt{n} \left(\left|\frac{(1 + \frac{1}{n})^2}{e^n}\right|\right) = \lim_{n \to \infty} \frac{1}{e} (1 + \frac{1}{n})^2 = \frac{1}{e} < 1 \]

Activity 3

Question 1: Use the method of infinitesimal approach to calculate area has delimited region by \( y = x^2 \), axis x, and \( x = -2 \) the lines \( x = 1 \);

Answer:

\[ \Delta_{x_i} = \frac{(1 - (-2))}{n} = \frac{3}{n} \]

\[ \Delta_{x_i} = X_{i} - X_{i-1} \]

\[ x_{i-1} = -2 + 2(i-1) \Delta_{x_i} = \frac{(3(i-1))}{N}, -2 \text{ to } i = 1, ..., n \]

In this second example - use is \( f(x_{i-1}) \) for subscribers rectangles

The \( \lim_{n \to \infty} \sum_{i=1}^{n} \left(x_{i-1}\right)^2 \Delta_{x_i} \)

\[ \lim_{n \to \infty} \sum_{i=1}^{n} \left((\frac{27}{n^2}) \sum_{i=1}^{n-1} i^2 \right) - \lim_{n \to \infty} \sum_{i=1}^{n} \left((\frac{36}{n^2}) \sum_{i=1}^{n-1} i \right) + \lim_{n \to \infty} \left(\frac{12}{n}\right) \cdot \frac{1}{n} \]

\[ = \lim_{n \to \infty} \left(\frac{27}{n^2}\right) \sum_{i=0}^{n-1} i^2 - \lim_{n \to \infty} \left(\frac{36}{n^2}\right) \sum_{i=0}^{n-1} i + \lim_{n \to \infty} \left(\frac{12}{n}\right) \cdot \frac{1}{n} \]

\[ = \lim_{n \to \infty} \left(\frac{27}{n^2}\right) \sum_{i=0}^{n-1} i^2 \]

\[ \lim_{n \to \infty} \left(\frac{27}{n^2}\right) \sum_{i=1}^{n-1} i^2 \cdot \lim_{n \to \infty} \left(\frac{36}{n^2}\right) \sum_{i=1}^{n-1} i + \lim_{n \to \infty} \left(\frac{12}{n}\right) \cdot \frac{1}{n} \]

\[ = \frac{9}{2} + 12 = 3 \]
Question 2: Find the exact area of the region bounded by \( y = (x^3 + 1) \), \( y = 0 \), \( x = -2 \) \( x = 2 \)

Answer:
\[
\int_{-2}^{2} (x^3 + 1) \, dx = \left[ \frac{x^4}{4} + x \right]_{-2}^{2} = \left( \frac{(2^4)}{4} + 2 \right) - \left( \frac{((-2)^4)}{4} + (-2) \right) = 4
\]

Question 3: Find the exact area of the region bounded by \( y = \sec x \), \( y = 0 \), \( x = -(\pi / 3) \) \( x = (\pi / 3) \)

Answer:
\[
\int_{-(\pi / 3)}^{(\pi / 3)} \sec x \, dx = \ln | \sec x + \tan x | \bigg|_{-(\pi / 3)}^{(\pi / 3)} = \ln \left( \frac{\sqrt{3} + 2}{2 - \sqrt{3}} \right) = 2.6339
\]
Question 4: Use the mean value theorem for integrals to prove the inequality $\int_{-(\pi/6)}^{(\pi/6)} \cos x^2 dx \leq \pi/3$

Answer: From the range $[-(\pi/6), (\pi/6)]$, the function lies between $(\sqrt{3}/2)$ and $\leq \cos x \leq 1$ $(3/4) \leq \cos^2 x \leq 1$ then the average value of the function must be between the same values:

$$(3/4) \leq \frac{\int_{-(\pi/6)}^{(\pi/6)} \cos x^2 dx}{((\pi/6) - (-\pi/6))} \leq 1 \Leftrightarrow \pi/4 \leq \int_{-(\pi/6)}^{(\pi/6)} \cos x^2 dx \leq \pi/3$$

Question 5: Find the average value of the function $f(x) = x^2$ given in the range $[-1,2]$, and give an $\chi$ point where $f(\chi)$ reaches that average.

Answer:

$$\int_{-1}^{2} x^2 dx = \left[ \frac{x^3}{3} \right]_{-1}^{2} = \left( \frac{2^3}{3} \right) - \left( \frac{(-1)^3}{3} \right) = \frac{8}{3} + \frac{1}{3} = 3$$

$$f(\chi) = \frac{3}{(2 - (-1))} = 1$$

$c^2 = 1 \Leftrightarrow \chi = 1$ to another solution $\chi = -1$ does not satisfy

Question 6: Calculate the derivatives of the mobile board integrals $\left( \frac{d}{dx} \int_{0}^{x} \frac{1}{t^4 + 4} dt \right)$

Answer: $\frac{1}{x^4 + 4}$

Question 7: Calculate the derivatives of the mobile board integrals $\left( \frac{d}{dx} \int_{0}^{\tan x} \frac{1}{1 + t^2} dt \right)$

Answer: $1$

Question 8: Calculate the area of the region bounded by the curves $y = x^2$, $x = y^3$ and $x + y = 2$

Answer:

Searching the intersections, substituting $y = 2$, $x$ in each equation

$$2 - x = x - 2 + x^2 \Leftrightarrow x = 0 \Leftrightarrow x = -2$$

$$x (2 - x) + 13x 6x^2 \Leftrightarrow x^3 - 8 - 1 \Leftrightarrow x = 0$$

The intersection of the abscissa is $x = 1$

$$\int_{0}^{1} \left( x^2 - x^3 \right) dx = \left[ \frac{x^3}{3} - \frac{(x^4)}{4} \right]_{0}^{1} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$
Question: Calculate the area of the region bounded by the curves
\[ y = \sin x \ ; \ Y = \cos x \text{ between two consecutive intersection points} .\]

Answer:
\[
\int_{\left( \frac{\pi}{4} \right)}^{\left( \frac{5\pi}{4} \right)} (\sin x \cos x) \, dx = - (\cos \left( \frac{5\pi}{4} \right) + \sin \left( \frac{5\pi}{4} \right)) + (\cos \left( \frac{\pi}{4} \right) + \sin \left( \frac{\pi}{4} \right)) = 2\sqrt{2}
\]

Module Summary

Applied calculus has a wide range of areas covered. R was characterized as a complete ordered field with the property of the Supreme. Any other set with the same properties is isomorphic to R. The finished line was defined, with the introduction of symbols - , + , and operations and exceptions to these symbols. Some concepts topological R relative to notable points and subsets of R were outlined. Several basic functions for its analytical expression in the independent variable, including polynomial, rational, with radical, logarithmic, exponential and logarithmic. Equations or inequalities that can arise when calculating the function domain were given. Sequences and subsequences were presented, recursive sequences and some examples of notable sequences.

Concept of continuous function was presented, the calculation functions of limits on the finished line R making use of the symbols + \( \infty \), -\( \infty \) to simplify the notation. Fifteen different types of limits have been presented a real function can have. Finally we were presented some theorems on limits of real functions.

The continuity of real functions of real variable were presented some local properties of continuous functions. Alternatively continuity characterization. Notable examples of continuous functions. Discontinuity functions. Global continuous functions properties (Theorem of Bolzano or intermediate value)


As for local and global extremes seen the theorems of Fermat, Weierstrass (revisits), Cauchy, Rolle, Lagrange and l’Hospital. Monotony functions and concavity graphics. Tracing function graph. Differential of a function, the Leibniz notation. Rudiments of differential equations of 1st order. This activity showed up the indispensable tools in the application of the calculation method: the primitives of elementary functions, the integral of a differential form, the fundamental theorem of calculus, integration techniques, the Riemann integral, the geometric interpretation of the integral Riemann and calculation of areas and volumes using elementary integration.

Also presented some applications such as the calculation area of plane figures, volume solids by cutting, circular disk and cylindrical shells, arc length, mass centroid moment and flat region.
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