Education: Mathematics MAT 11

ANALYSIS 2

Jairus M. Khalagai
Foreword

The African Virtual University (AVU) is proud to participate in increasing access to education in African countries through the production of quality learning materials. We are also proud to contribute to global knowledge as our Open Educational Resources (OERs) are mostly accessed from outside the African continent. This module was prepared in collaboration with twenty one (21) African partner institutions which participated in the AVU Multinational Project I and II.

From 2005 to 2011, an ICT-integrated Teacher Education Program, funded by the African Development Bank, was developed and offered by 12 universities drawn from 10 countries which worked collaboratively to design, develop, and deliver their own Open Distance and e-Learning (ODEL) programs for teachers in Biology, Chemistry, Physics, Math, ICTs for teachers, and Teacher Education Professional Development. Four Bachelors of Education in mathematics and sciences were developed and peer-reviewed by African Subject Matter Experts (SMEs) from the participating institutions. A total of 73 modules were developed and translated to ensure availability in English, French and Portuguese making it a total of 219 modules. These modules have also been made available as Open Educational Resources (OER) on oer.avu.org, and have since then been accessed over 2 million times.

In 2012 a second phase of this project was launched to build on the existing teacher education modules, learning from the lessons of the existing teacher education program, reviewing the existing modules and creating new ones. This exercise was completed in 2017.

On behalf of the African Virtual University and our patron, our partner institutions, the African Development Bank, I invite you to use this module in your institution, for your own education, to share it as widely as possible, and to participate actively in the AVU communities of practice of your interest. We are committed to be on the frontline of developing and sharing open educational resources.

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The following institutions participated in the teacher education program of the Multinational Project I: University of Nairobi – Kenya, Kyambogo University – Uganda, Open University of Tanzania, University of Zambia, University of Zimbabwe – Zimbabwe, Jimma University – Ethiopia, Amoud University - Somalia; Université Cheikh Anta Diop (UCAD)-Senegal, Université d’ Antananarivo – Madagascar, Universidade Pedagogica – Mozambique, East African University - Somalia, and University of Hargeisa - Somalia.
The following institutions participated in the teacher education program of the Multinational Project II: University of Juba (UOJ) - South Sudan, University of The Gambia (UTG), University of Port Harcourt (UNIPORT) – Nigeria, Open University of Sudan (OUS) – Sudan, University of Education Winneba (UEW) – Ghana, University of Cape Verde (UniCV) – Cape Verde, Institut des Sciences (IDS) – Burkina Faso, Ecole Normale Supérieure (ENSUP) - Mali, Université Abdou Moumouni (UAM) - Niger, Institut Supérieur Pédagogique de la Gombe (ISPG) – Democratic Republic of Congo and Escola Normal Superior Tchicote – Guinea Bissau

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This second edition is the product of a review process based on the first edition of this module. The information provided below, except the author of first edition, refers to the second edition.

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Introduction

Prerequisites

Unit 1: Topology
Analysis 1 (unit 1) and Unit (4)

Unit 2: Measure Theory
Analysis 1 (unit 1)

Unit 3: Lebesgue integral
(unit 6)

Time
The total time for this module is 120 study hours.

Materials
Students should have access to the core readings specified later. Also, they will need a computer to gain full access to the core readings. Additionally, students should be able to install the computer software wxMaxima and use it to practice algebraic concepts.

Module Rationale
The rationale of teaching analysis is to set the minimum content of Pure Mathematics required at undergraduate level for student of mathematics. It is important to note that skill in proving mathematical statements is one aspect that learners of Mathematics should acquire. The ability to give a complete and clear proof of a theorem is essential for the learner so that he or she can finally get to full details and rigor of analyzing mathematical concepts. Indeed it is in Analysis that the learner is given the exposition of subject matter as well as the techniques of proof equally. We also note here that if a course like calculus with its wide applications in Mathematical sciences is an end in itself then Analysis is the means by which we get to that end.

Overview
This module consists of three units which are as follows:

Unit 1: Topology
The structures of topological spaces are studied along side those of metric spaces. However, the main difference here is that the axioms that define a metric space are dependent on the concept of distance whereas in the axioms that define a topological space the concept of distance is absent. In particular the study of the twin concepts of convergence and continuity brings out this difference very well. Finally a look at different topologies like product or quotient topology endowed on a set is essential in this unit.
Unit 2: Measure Theory

In this unit we start with the study of both the Lebesgue outer measure and the real line before we look at the Lebesgue measurable subsets of the real line. A look at the sigma algebra of subsets of a given underlying set gives rise to measurable space on which we can also study a class of functions called measurable functions. A part from the Lebesgue measure we also study an abstract measure leading to an abstract measure space on which we introduce an abstract integral.

Unit 3: Lebesgue integral

In this unit we define the Lebesgue integral giving examples of Lebesgue integrals defined on different sets. State some properties of Lebesgue integral, and verify some properties of the Lebesgue integral. Finally a brief comparison of the Lebesgue integral and the well known Riemann integral is also essential in this unit.

Outline

Unit 1: Topology (40 hours)

Level priority Unit 1 is the Pre-requisite

- Review of metric spaces
- Topological spaces, neighbourhoods, interior and open sets
- Limit points closed sets, closure and boundary
- Bases, relative and product topologies
- Continuity and homeomorphisms
- Convergence and Hansdorff axiom.

Unit 2: Measure Theory (40 hours)

Level priority Unit 2 are the Pre-requisite

- Lebesgue outer measure and Lebesgue measure on real line
- Lebesgue measurable subsets of
- Measurable spaces and measurable functions
- Abstract measure spaces and abstract integral
- Monotone convergence theorem, Fatou’s lemma and Lebesgue dominated convergence theorem
- Relation between Riemann Integral and Lebesgue Integral
Unit 2: The Lebesgue Integral (40 hours)

Level priority Unit 3 is the Pre-requisite

- Define the Lebesgue integral
- Give examples of Lebesgue integrals defined on different sets.
- State some properties of Lebesgue integral
- Verify some properties of the Lebesgue integral

Graphic Organizer

Specific Learning Objectives (Instructional Objectives)

You should be able to:

1. Demonstrate understanding of basic concepts and principles of mathematical analysis.
2. Develop a logical framework for stating and proving theorems.
Pre-Assessment

1. Which of these sets is countable?
   - (a) $\mathbb{R}$
   - (b) $\mathbb{Q}^C$
   - (c) $[0,1]$
   - (d) $A = \mathbb{Q} \cap [a,b]$

   where $\mathbb{Q}$ is the set of rational numbers.

2. Which of these sets has both supremum and infimum
   - (a) $A = \{ x : 0 < x < 1 \}$
   - (b) $B = \{ x : -\infty < x \leq 1 \}$
   - (c) $C = \{ x : x > 1 \}$
   - (d) $D = \{ x : x < 1 \}$

3. Given the set $A = \left\{ \frac{1}{n}, n \in \mathbb{N}^+ \right\}$ how many limit points does $A'$ have?
   - (a) 2
   - (b) 0
   - (c) 1
   - (d) Infinite

4. Given the sequence of real numbers $x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ which of the following statements is true about the sequence $(x_n)$?
   - (a) $x_n$ converges to 1
   - (b) $x_n$ converges to 0
   - (c) $x_n$ is bounded
   - (d) $x_n$ is Cauchy

5. Which of the following subsets of $\mathbb{R}$ is an open set?
   - (a) $A = \{ x : 1 < x \leq 2 \}$
   - (b) $B = \{ x : 0 < x < 1 \}$
   - (c) $C = \{ x : 0 \leq x < 2 \}$
   - (d) $D = \{ x : 1 \leq x \leq 2 \}$
6. Which of the following subsets of \( \mathbb{R} \) is a closed set?
   
   (a) \( A = \{ x : 1 \leq x \leq 5 \} \)
   
   (b) \( B = \{ x : 20 < x < 30 \} \)
   
   (c) \( C = \{ x : 0 < x \leq 1 \} \)
   
   (d) \( D = \{ x : 6 > x \geq 0 \} \)

7. State which of the following subsets of \( \mathbb{R} \) is neither open nor closed.
   
   (a) \( A = \{ x : 2 \leq x \leq 5 \} \)
   
   (b) \( B = \{ x : 0 < x < 3 \} \)
   
   (c) \( C = \{ x : 0 < x \leq 4 \} \)
   
   (d) \( D = \{ x : 0 < x \leq 4 \} \)

8. Which of the following statements about a function of bounded variation is true?
   
   (a) It is continuous
   
   (b) It is discontinuous
   
   (c) It is uniformly continuous
   
   (d) It is bounded

9. The following is a Riemann sum: \( \frac{\pi}{n} \sum_{k=1}^{n} \sin \frac{k\pi}{n} \). Use this fact to evaluate
   
   \( \lim_{n \to \infty} \left( \frac{\pi}{n} \sum_{k=1}^{n} \sin \frac{k\pi}{n} \right) \)
   
   (a) 2
   
   (b) 0
   
   (c) -2
   
   (d) -1
Answers

1. D
2. A
3. C
4. C
5. B
6. A
7. C
8. D
9. A
Unit 1: Topology

Specific Objectives

At the end of this activity the learner will be able to:

- Define a topology on a set
- Give examples of topological spaces
- Recast the definitions of the components of the structure like neighbourhoods, interior point, limit point etc without reference to the concept of distance
- Recast definition of continuity of a function without reference to the concept of distance
- State and prove properties of continuity purely in terms of subsets of the space

Summary

We summarise this activity along the same lines as in activity 2 in which we saw that structure of a metric space is heavily depended on the concept of distance. In the case of a topological space (see definition in the next section) the structure depends on both open and closed sets.

Thus, our main task in this activity is to see how we can recast those definitions in the metric space without making any reference to the concept of distance since we note that in a set based structure the concept of distance is generally absent. It is also important to note that any definition in a metric space which does not refer to distance can still carry through in a general topological space.

We also note that the concept of continuity of functions in particular acquires considerable generalization when its definition and properties are stated in terms of open sets. This is significant since the concept of continuity has exceptional properties which carry a lot of applications in mathematics.

Key Terms

**Topology:** Let $X$ be any non-empty set and $\tau$ be a collection of subsets of $X$. Then $\tau$ is called a topology on $X$ if the following properties are satisfied

(i) $\emptyset \in \tau$
(ii) $X \in \tau$
(iii) $\bigcap_{i=1}^{n} O_i \in \tau$ whenever $O_i \in \tau \quad \forall \ i = 1, \ldots, n.$
(iv) $\bigcup_{\alpha \in \Omega} O_\alpha \in \tau$ whenever $O_\alpha \in \tau$ for each $\alpha \in \Omega$.

**Open set:** Let $A$ be a subset of a topological space $(X, \tau)$, then $A$ is said to be open if $A \in \tau$.

**Closed set:** Let $A$ be a subset of a topological space $(X, \tau)$, then $A$ is said to be closed if its
complement denoted by \( C(A) \in \tau \).

**Neighbourhood:** Let \((X, \tau)\) be a topological space and \( p \in X \). Then a subset \( N \) of \( X \) is called a neighbourhood of \( p \) if \( N \) contains an open set \( O \) which contains \( p \). i.e. \( p \in O \subset N \) for \( O \in \tau \). In this case \( p \) is also called an interior point of the subset \( N \).

**Interior of a set:** Let \( A \) be a subset of a topological space \((X, \tau)\), then interior of \( A \) denoted by \( A^0 \) or \( \text{Int} \, A \) is the set of all interior points of \( A \). Clearly \( \text{Int} \, A \subseteq A \).

**Limit Point:** Let \((X, \tau)\) be a topological space and \( A \) be a subset of \( X \). Then a point \( x \in X \) is said to be a limit point of \( A \) if for each neighbourhood \( N \) of \( x \) we have that \( N \cap A \neq \emptyset \).

**Closure of a set:** For any set \( A \) in \((X, \tau)\) closure of \( A \) denoted by \( \overline{A} \) is given by \( \overline{A} = A \cup \{ x : x \text{ is a limit point of } A \} \). Clearly \( A \subseteq \overline{A} \).

**Boundary of a set:** For any subset \( A \) of a topological space \( X \), boundary of \( A \) denoted by \( B' \text{dary} \, (A) = \overline{A} \cap C(A) \).

**Continuity of a function:** Let \((X, \tau_1)\) and \((Y, \tau_2)\) be two topological spaces and \( f : (X, \tau_1) \to (Y, \tau_2) \) be a function. Then \( f \) is said to be continuous on \( X \) if \( f^{-1}(0) \) is open in \( X \) for every open set \( O \) in \( Y \). Where \( f^{-1}(0) = \{ x \in X : f(x) \in O \} \). Thus \( f \) is continuous if \( f^{-1}(0) \in \tau_1 \) whenever \( O \in \tau_2 \).

**Homeomorphism:** Let \( X \) and \( Y \) be topological spaces. If both \( f : X \to Y \) and its inverse \( f^{-1} : Y \to X \) are continuous then \( f \) is called a homeomorphism.

### Activity 1: On Set Based Structure of a Space.

**Story of a designer**

One day while walking in the streets of Nairobi, the Capital City of Kenya I came across a designer who was using a drawing package (OpenOffice Draw) to create some symmetrical images. She was doing this in stages whereby she first drew a star then produced about a hundred copies of them. She then assembled the copies into a symmetrical image.

**Stage 1**
Stage 2

Question

Can you make a suggestion of any other shape whose copies can be assembled to create a symmetrical pattern?

Introduction

We note that in our learning activity 2, we referred to some geometric structure which accounts for symmetry or beauty. However, in our story above we have a case where beauty or symmetrical images can be created without necessarily using the concept of distance.

The structure here depends on some arrangement of collections of items in the space. This is a situation that is analogous to the structure of a general topological space which is dependent on sets and their collections. Indeed the definitions in a topological space as we have already seen depend on components like neighbourhoods, open sets or closed sets or their collections.

It is also important to note that the definitions in metric spaces which do not refer to the concept of distances will remain unchanged in general topological spaces. For instance definitions like, open cover, compact set etc are intact whether in metric space or topological space.

In this connection the set based structure on a topological space re-organizes the sequence of concepts in such a way that can be summarized by the following diagram.
Evaluation

Example

The following are examples of general topological spaces.

Let $X = \{a, b, c\}$. Define

- $\tau_1 = \{\emptyset, X, \{a, b\}, \{b\}, \{a\}, \{b, c\}\}$
- $\tau_2 = \{\emptyset, X\}$
- $\tau_3 = \{\text{All subsets of } X\}$

Then $\tau_1, \tau_2, \text{and } \tau_3$ are all topologies on $X$. Where $\tau_3$ is called the finest topology on $X$ and $\tau_2$ is called the coarsest topology on $X$. Thus $(X, \tau_1), (X, \tau_2), \text{and } (X, \tau_3)$ are topological spaces.

Consider $X = \mathbb{R}$ with its standard metric an let

- $\tau = \{\text{All open intervals in } \mathbb{R}\}$. Then $\mathbb{R}$ is a topological space. Let $X$ be any metric space and let

- $\tau = \{\text{All open subsets of } X\}$

Then $(X, \tau)$ is a topological space.

In order for the learner to understand example 5.1 part (iii) better we now work out the following exercise together.

Exercise

In any metric space $X$ show that:

The null set $\emptyset$ is open

The whole space $X$ is open

$\bigcap_{i=1}^{n} O_i$ is open whenever $O_i$ is open $\forall i = 1, \ldots, n$

$\bigcup_{\alpha \in \Omega} O_\alpha$ is open whenever $O_\alpha$ is open for each $\alpha \in \Omega$.

Solution

If $\emptyset$ is not open then $\exists x \in \emptyset$ which is not an interior point. But there is no such $x$ in $\emptyset$ since $\emptyset$ is empty. This is a contradiction. Hence $\emptyset$ is open.

For any point $p \in X$ $\exists r > 0$ such that the neighbourhood $N(p, r) \subseteq X$. Thus any point $p \in X$ is an interior point. Hence $X$ is open.

Assume $O_1, O_2, \ldots, O_n$ are open.

Let $O = \bigcap_{i=1}^{n} O_i$ and $p$ be any point in $O$. Thus $p \in O$ implies $p \in O_i \forall i = 1, \ldots, n$.

Thus $\exists$ nbhs $N(p, r_i)$ such that $N(p, r_i) \subseteq O_i \forall i = 1, \ldots, n$. 
Let \( r = \min_{1 \leq i \leq n} \{ r_i \} \). Then the nbh \( N(p, r) \) is such that:

\[
p \in N(p, r) \subset O = \bigcap_{i=1}^{n} O_i
\]

Hence \( O = \bigcap_{i=1}^{n} O_i \) is open.

Let \( O_\alpha \) be open for each \( \alpha \in \Omega \) and let \( O = \bigcup_{\alpha \in \Omega} O_\alpha \).

Now \( p \in O \Rightarrow p \in O_\alpha \) for some \( \alpha \in \Omega \). But \( O_\alpha \) is open. Hence \( \exists \) a nbh say \( N(p, r) \) of \( p \) such that

\[
p \in N(p, r) \subset O_\alpha \subset \bigcup_{\alpha \in \Omega} O_\alpha
\]

i.e. \( p \) is an interior point of \( O \).

Hence \( O = \bigcup_{\alpha \in \Omega} O_\alpha \) is open.

**Remark**

We note from the solution of the exercise above that if \( \tau \) is the collection of all open subsets of a metric space \( X \) then we have that:

(i) \( \emptyset \in \tau \)

(ii) \( X \in \tau \)

(iii) \( \bigcap_{i=1}^{n} O_i \in \tau \), whenever \( O_i \in \tau \) \( \forall \ i = 1, \ldots, n \).

(iv) \( \bigcup_{\alpha \in \Omega} O_\alpha \in \tau \), whenever \( O_\alpha \in \tau \) for each \( \alpha \in \Omega \).

This is why every metric space is an example of a topological space.

Now try the following exercise.

**Exercise**

1. In any metric space \( X \) show that:

   (i) \( \emptyset \) is closed

   (ii) \( X \) is closed

   (iii) \( \bigcup_{i=1}^{n} F_i \) is closed whenever \( F_i \) is closed \( \forall \ i = 1, \ldots, n \).

   (iv) \( \bigcap_{i=1}^{n} F_\alpha \) is closed whenever \( F_\alpha \) is closed for each \( \alpha \in \Omega \).

2. Give reasons why we can not use the collection of closed sets to define a topology on a metric space \( X \).

We now state some theorems which give us deeper relationship among some concepts in a topological space.
**Theorem**

Let $X$ be a topological space. Then we have:

For each $x \in X$, $\exists$ a neighbourhood $N$ of $x$ in $X$.

If $N$ is a neighbourhood of $x \in X$ then $x \in N$.

If for each, neighbourhood $N$ of $x$ we have $N' \supset N$, then $N'$ is also a neighbourhood of $x$.

If for each $x \in X$ both $N$ and $M$ are neighbourhoods of $x$ then $N \cap M$ is also a neighbourhood of $x$.

**Theorem**

Let $X$ be a topological space. Then a subset $O$ of $X$ is open iff $O$ is a neighbourhood of each of its points.

**Theorem**

Given a subset $A$ of a topological space $X$ we have:

If $F$ is a closed set such that $A \subseteq F$, then $\overline{A} \subseteq F$.

If $\{ F_\alpha \}_{\alpha \in \Omega}$ is a family of closed sets with each one of them containing $A$, then $\overline{A} = \bigcap_{\alpha \in \Omega} F_\alpha$.

If $\{ O_\alpha \}_{\alpha \in \Omega}$ is a family of open subsets of $X$ with each one of them contained in $A$ then $\text{Int} \ A = \bigcup_{\alpha \in \Omega} O_\alpha$.

**Remark**

Note that part (i) of the theorem above simply asserts that “every closed set $F$ that contains $A$ also contains closure of $A$” while part (ii) simply says that: “Closure of $A$ is the smallest closed set that contains $A$”. We also note that part (iii) asserts that interior of a set is the largest open subset of the set.

**Exercise**

Using the readings try to give the proof of theorem 5.6 above.
Remark

Note that the definition of continuity on a topological space can also be stated in terms of closed sets. Thus \( f : X \to Y \) is continuous if \( f^{-1}(A) \) is closed in \( X \) whenever \( A \) is closed in \( Y \).

It is in view of this equivalent definition that the theorem stated below can be proved.

Theorem

Let \( X \) and \( Y \) be topological spaces and \( f : X \to Y \) be a function. Then \( f \) is continuous iff for any subsets \( A \) of \( X \) we have

\[
f(\overline{A}) \subseteq \overline{f(A)}
\]

Exercise

1. For any two subsets \( A \) and \( B \) of a topological space \( X \) show that
   
   (i) \( \overline{A \cup B} = \overline{A} \cup \overline{B} \)
   
   (ii) \( \overline{A \cap B} \subseteq \overline{A} \cap \overline{B} \)

2. By expressing closure of a set \( A \) as: \( \overline{A} = A \cup B' \) dary \( (A) \),
   show that: \( A \) is closed if \( B \) dary \( (A) \) \( \subseteq \) \( A \)

3. Make a brief comparison between continuity of a function in metric spaces and continuity of a function in general topological spaces.

4. Consider the open interval \((0,1)\) with any other open interval say \((a,b)\) in \( \mathbb{R} \).
   Let \( X = (0,1) \) and \( Y = (a,b) \) in \( \mathbb{R} \) be endowed with the standard topology.
   Let also \( f : X \to Y \) be defined by \( f(t) = a(1-t) + bt \) \( \forall \ t \in (0,1) \)
   Show that \( f \) is a homeomorphism from \( X \) onto \( Y \).
Unit 2: Measure Theory.

Specific Objectives

At the end of this activity the learner will be able to:

- Give definition of the Lebesgue outer measure and state its properties.
- Give definition and examples of Lebesgue measurable sets.
- Define and give examples of \(\sigma\)-algebras.
- Define and give examples of measurable spaces.
- Define and give examples of measurable functions.
- Define and give examples of measure spaces.

Summary

We have already seen in the last two activities that a space is a non-empty set endowed with either geometric or algebraic structure. Indeed, in the case of a metric space \((X, d)\) we have a geometric structure since \(d\) is distance while in the case of a topological space \((X, \tau)\) we have an algebraic structure since the analysis in \(X\) depends on the collection \(\tau\) of sets and their combinations. In this activity we look at a measurable space along the same lines and then extend this to the concept of a measure space.

We also define a class of functions called measurable functions and study their properties in order to understand the structure in a deeper sense. We note here that like the case of a topological space, the structure of a measurable space also depends on collections of sets. However, these collections of sets are such that we can introduce a measure on them. This in itself suggests that the structure of a measure space is both geometric and algebraic since in a measure we have the concept of length.

Key Concepts

**Lebesgue Outer Measure:** Let \(\lambda^*(I)\) denote the length of an interval \(I\) and \(\lambda^*(r) = \sum_{I \in r} \lambda(I)\) where \(r\) is a countable collection of open subintervals of \(\mathbb{R}\). For any subset \(E\) of \(\mathbb{R}\) we denote the class of all countable collections of open subintervals of \(\mathbb{R}\) which cover \(E\) by \(C(E)\). Then the number given by \(\mu^*(E) = \inf_{r \in C(E)} \lambda^*(r)\) is called Lebesgue outer measure of the set \(E\).

**Lebesgue measurable set:** Any subset \(E\) of \(\mathbb{R}\) is said to be Lebesgue measurable if for any other subset \(Y\) of \(\mathbb{R}\) we have that: \(\mu^*(Y) = \mu^*(Y \cap E) + \mu^*(Y \cap E^C)\)
**Algebra:** Let $X$ be a non-empty set and $\mathcal{X}$ be a collection of subsets of $X$ such that:

**Measurable space:** Let $X$ be a non-empty set and $\mathcal{X}$ denote the $\sigma$–algebra of all subsets of $X$. Then the ordered pair $(X, \mathcal{X})$ is called a measurable space.

Thus any member of $\mathcal{X}$ is called $\mathcal{X}$-measurable set.

**Measurable function:** Let $(X, \mathcal{X})$ be a measurable space and $f : X \to \mathbb{R}$ be an extended real valued function. Then $f$ is said to be $\mathcal{X}$-measurable if the set: 

$$\{ x \in X : f(x) < r \} \in \mathcal{X} \quad \forall r \in \mathbb{R}.$$ 

**Measure:** Let $(X, \mathcal{X})$ be a measurable space. The restriction of the outer measure to only members of $\mathcal{X}$ is called a measure.

**Measure Space:** Let $(X, \mathcal{X})$ be a measurable space and $\mu$ be a measure on $(X, \mathcal{X})$. Then the ordered triple $(X, \mathcal{X}, \mu)$ is called a measure space.

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**Activity 1: On Set Based Structure with Measures.**

**Story of Designing Symmetrical Patterns**

In our activity 3 we had a story of a designer who could create some symmetrical images by simply generating patterns without reference to distance or any measurements. In this story we draw your attention to the fact that there are times when designers also require measurements in order to create symmetrical images. While walking on the streets of Nairobi one day I also came across such a designer who would start with say a number of squares on a line then measure some distance and angle before putting other figures like equilateral triangles on a line e.t.c. until a complete figure like a hexagon is created.
Question

Apart from generating a hexagon whose sides has different shapes of figures can you generate other patterns?

Introduction

We note that in the story above the designer is not just concerned with producing copies of a given figure like a star but has also to use measurements of length and angles to generate a pattern as seen in the diagram above.

This is a case which best relates to a structure that combines both geometry and algebra in a space. In Measure Theory we deal with a space called Measure Space (as already defined in previous section) where we have the ordered triple \( (X, \mathcal{F}, \mu) \). We note that apart from the underlying set \( X \) itself, the \( \sigma \)– algebra \( \mathcal{F} \) is about combinations of subsets of \( X \) which is the algebraic aspect of the space. However the measure \( \mu \) is about measurements on the subsets of \( X \). For example if \( I \) is an interval in \( \mathbb{R} \), then \( \mu(I) \) is simply the length of the interval \( I \). Thus \( \mu \) constitutes the geometric aspect of the structure of a measure space. It now follows that the definitions of concepts in a measure space have to depend on these basic components of the structure in the space. In fact we can summarise the sequence of concepts in this space as follows.
Example

1. Let $\mathcal{X} = \mathbb{R}$, the class of all Lesbegue measurable subsets of $\mathbb{R}$ is denoted by $\mathcal{m}$ and we have that the null set $\phi$, $\mathbb{Q}$ and $\mathbb{R}$ belong to $\mathcal{m}$. Also $\mathcal{m}$ is a $\sigma -$ algebra of subsets of $\mathbb{R}$.

2. Let $\mathcal{P}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R})$ denote the class of all subsets of $\mathbb{R}$ and the class of the Borel subsets of $\mathbb{R}$.

Then $\mathcal{P}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R})$ are also $\sigma -$ algebra and we have that:

$$\mathcal{B}(\mathbb{R}) \subset \mathcal{m} \subset \mathcal{P}(\mathbb{R}).$$

Note that we also have that:

$$\mu = \mu^* / \mathcal{m}$$

is the Lebesgue measure on $\mathbb{R}$.

3. Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{m} = \mathcal{P}(\mathbb{R})$ or $\mathcal{m} = \mathcal{m}$ or $\mathcal{m} = \mathcal{B}(\mathbb{R})$, then we have that $\mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathbb{R})$, and $\mathcal{B}(\mathbb{R})$ are measurable spaces.

4. Let $\mathcal{X} = \mathbb{R}$, $\mathcal{m} = \mathcal{m}$ and $\mu = \mu^* / \mathcal{m}$ be the Lebesgue measure on $\mathbb{R}$.

Then the ordered triple $(\mathbb{R}, \mathcal{m}, \mu)$ is a measure space.

Example

Let $E$ be a measurable subset of $X$.

Consider the function $\chi_E$ defined by:

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in E^C \end{cases}$$

We show that $\chi_E$ is measurable, where $\chi_E$ is called the characteristics function of the set $E$.

Indeed, for $r > 1$, the set: $\{ x \in X : \chi_E(x) < r \} = X \in \mathcal{X}.$

For $0 < r \leq 1$, the set:

$$\{ x \in X : \chi_E(x) < r \} = E^C \in \mathcal{X}.$$

Also for $r \leq 0$, the set:

$$\{ x \in X : \chi_E(x) < r \} = \phi \in \mathcal{X}.$$

Hence for all $r \in \mathbb{R}$ we have that the set:

$$\{ x \in X : \chi_E(x) < r \} \in \mathcal{X}.$$

Thus $\chi_E$ is measurable.

Note that the class of all measurable functions on a measurable functions on a measurable space $(X, \mathcal{X})$ is denoted by $M(X, \mathcal{X})$. 
In this case \( \chi_E \in M(X, \mathcal{A}) \).

If \( X = \mathbb{R} \) and \( \mathcal{A} = \sigma \mathbf{m} \) then we say that \( \chi_E \) is a Lebesgue Measurable function.

**Exercise**

In the example above letting \( X = \mathbb{R} \) and \( \mathcal{A} = \sigma \mathbf{m} \) sketch the graph of \( \chi_E \) and use it to verify that \( \chi_E \) is indeed Lebesgue measurable.

We now state some theorems which give us properties and some relationship among some concepts in measurable spaces and measure spaces.

**Theorem**

Let \( \mu^* \) be the Lebesgue outer measure on the subsets of \( \mathbb{R} \). Then we have:

(i) \( \mu^*(\emptyset) = 0 \)

(ii) \( \mu^*(\{x\}) = 0 \quad \forall \ x \in \mathbb{R} \)

(iii) \( \mu^* \) is a monotone i.e. \( \mu^*(A) \leq \mu^*(B) \) for \( A \subseteq B \)

(iv) \( \mu^* \) is translation invariant. i.e. \( \mu^*(\tau_r(E)) = \mu^*(E) \)

where \( \tau_r(x) = x + r \quad \forall \ x \in \mathbb{R} \)

(v) \( \mu^* \) is countably sub-additive i.e. if \( (E_n)_{n=1}^{\infty} \) is any sequence of subsets of \( \mathbb{R} \) then

\[
\mu^* \left( \bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)
\]

**Remarks**

Note that apart from the properties stated above we have from definition of

\[
\mu^*: \mathcal{P}(\mathbb{R}) \to \mathbb{R}^+_0 \quad \text{that} \quad \mu^* \geq 0.
\]

Note also that the main difference between the properties of the Lebesgue outer measure \( \mu^* \) on \( \mathbb{R} \) and the Lebesgue measure \( \mu \) on \( \mathbb{R} \) is that \( \mu \) is countably additive. Thus if \( (E_n)_{n=1}^{\infty} \) is a sequence of subsets of \( \mathbb{R} \) then

\[
\mu \left( \bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n)
\]
**Theorem**

Let \((X, \mathcal{B})\) be a measurable space and \(f, g : X \to \mathbb{R}^e\) be \(\mathcal{B}\) measurable. Let \(c\) be any constant. Then we have:

(i) \(f + g\) is \(\mathcal{B}\) measurable
(ii) \(c + f\) is \(\mathcal{B}\) measurable
(iii) \(cf\) is \(\mathcal{B}\) measurable

**Exercise**

(a) Let \(E\) be a countable set in the sense that we can arrange it as:
\[E = \{x_1, x_1, \ldots\}\]
Show that \(\mu^*(E) = 0\)
Show that for the set \(\mathbb{Q}\) of rational numbers \(\mu^*(\mathbb{Q}) = 0\).

From the definition of a \(\sigma\) - algebra \(\mathcal{B}\), show that

(i) \(X \in \mathcal{B}\)
(ii) \(\bigcap_{n=1}^{\infty} A_n \in \mathcal{B}\)

Let \((X, \mathcal{B}), (Y, \mathcal{Y})\) be two measurable spaces and \(f : X \to Y\) be a function. Give definition of measurability of \(f\) along the lines of definition of continuity in topological spaces.

Show that if \(f : \mathbb{R} \to \mathbb{R}\) is continuous then \(f\) is Lebesgue measurable.

(i) Show that if a function \(f\) is measurable then \(|f|\) is also measurable.
(ii) Let \(f : A \to \mathbb{R}\) where \(A \notin \sigma m\) be given by
\[f(x) = \begin{cases} 1 & \forall x \in A \\ -1 & \forall x \in A^C \end{cases}\]
Show that \(|f|\) is Lebesgue measurable but \(f\) is not.

**Readings**

Unit 3: The Lebesgue Integral

Specific Objectives

At the end of this activity the learner will be able to:

- Define the Lebesgue integral
- Give examples of Lebesgue integrals defined on different sets.
- State some properties of Lebesgue integral
- Verify some properties of the Lebesgue integral

Summary

We have already seen in unit one of Analysis 1 Module how the Riemann integral is developed and the restrictions on the nature of the functions that are Riemann integrable. Indeed Riemann integrable functions are subject to rather stringent continuity conditions. However the Lebesgue theory enables us to integrate a much larger class of functions. Its greatest advantage lies perhaps in the ease with which many limit operations can be handled and from this point of view the Lebesgue convergence theorems may well be regarded as the core of the Lebesgue theory.

One of the difficulties which is encountered in the Riemann theory is that limits of Riemann integrable functions (or even of continuous functions) may fail to be Riemann integrable. This difficulty is eliminated in Lebesgue theory since limits of measurable functions are always measurable.

Key Terms

**Simple function:** Let \( f : X \rightarrow \mathbb{R} \) be a function whose range is finite. Then \( f \) is called a simple function. For example a constant function is a simple function.

**Canonical representation of a simple function:**
Suppose the range of a simple function \( f \) consists of the distinct non zero real numbers \( a_1, a_2, \ldots, a_n \). Let
\[
 f^{-1}(a_i) = E_i \quad \forall i = 1, \ldots, n, 
\]
so that
\[
 f(x) = a_i \quad \forall x \in E_i.
\]
Then \( E_i \cap E_j = \emptyset \) for \( i \neq j \) and the simple function \( f \) has the canonical representation as:
\[
 f = \sum_{i=1}^{n} a_i \chi_{E_i}.
\]
where \( \chi_{E_i} \) is the characteristic function on 
\( E_i \quad \forall i = 1, \ldots, n. \)

**Integral of a simple function:** Let \( (X, \mathcal{F}, \mu) \) be a measure space and let \( Q \) be a simple function in \( M^+ (X, \mathcal{F}) \). Thus \( Q \) is non-negative \( \mathcal{F} \) measurable simple function. Let \( Q \) have the canonical representation as

\[
Q = \sum_{k=1}^{n} a_k \chi_{E_k}.
\]

Then the extended real number given by

\[
\sum_{k=1}^{n} a_k \mu(E_k)
\]

is called the integral of the simple function \( Q \) with respect to \( \mu \) and is denoted by

\[
\int_X Q \, d\mu
\]

**Abstract integral:** Let \( (X, \mathcal{F}, \mu) \) be a measure space and \( f : X \to \mathbb{R}_e \) be any function in \( M^+ (X, \mathcal{F}) \). Then the integral of \( f \) denoted by \( \int_X f \, d\mu \) is given by

\[
\int_X f \, d\mu = \sup \left( \int_X S \, d\mu \right)
\]

where the supremum is taken over all simple functions \( S \) satisfying the condition

\[
S(x) \leq f(x) \quad \forall x \in X.
\]

If \( E \in \mathcal{F} \), then the integral \( \int_E f \, d\mu = \int_X f \chi_E \, d\mu \)

If \( \sup_s \int s \, d\mu = +\infty \) then \( \int f \, d\mu = +\infty. \)

If \( \mu \) is the Lebesgue measure then \( \int f \, d\mu \) is called the Lebesgue integral of the function \( f \).
Activity 1: The Lebesgue Integral

Story of felling a tree

It is a story told in a village called Luyia of Transzoia district in Kenya that about fifty years ago a man called Mulunda wanted to build a house. He took only a panga to try and fell a big tree for this purpose. One passerby just laughed at him wondering why Mulunda had embarked on an almost impossible mission. The second passerby sympathised with Mulunda so much that he went and borrowed an axe for him. While using an axe Mulunda realized that it took him a short time to fell the tree. He also admitted his ignorance of some neighbours owning tools like axes which can make work easier.

Question

Apart from an axe can you think of other tools that can be used to fell a big tree?

Introduction

We note that the main message from the story above is that Mr. Mulunda had owned a panga for a long time over trusting it with every work without being innovative enough to think of other tools. Indeed he only needed to consult with some of his neighbours to know that some jobs are beyond certain tools. It is significant to note that mathematical tools also behave in a similar manner. There are concepts which are beyond certain techniques. Indeed the Lesbegue theory enables us to evaluate integrals of a large class of functions than the Riemann theory would do. It is a well known fact that the limit of a Riemann integrable function need not be Riemann integrable. However, in the case of the Lebesgue integral the limit of a Lebesgue integrable function is still integrable.

We now state without proofs some of the most important theorems on monotonicity and convergence of the Lebesgue integral.

Theorem

Let \((X, \mathcal{F}, \mu)\) be a measure space. Then we have:

If \(f, g \in M^+(X, \mathcal{F})\) such that \(f \leq g\) then \(\int f \, d\mu \leq \int g \, d\mu\)

If \(E, f \in \mathcal{F}\) with \(E \subseteq F\) then \(\int_E f \, d\mu \leq \int_F f \, d\mu\)
Remark

Note that the result above shows that the Lebesgue integral is monotonic with respect to functions and sets.

Theorem 7.3 (Monotonic convergence theorem)

Let \((X, \mathcal{F}, \mu)\) be a measure space and \((f_n)\) be a sequence of functions in \(M^+(X, \mathcal{F})\) such that \((f_n)\) is monotonic increasing and converges to a function \(f \mu\text{-a.e.}\) on \(X\). Then we have:

\[
\lim_{n \to \infty} \int f_n \, d\mu = \int \left( \lim_{n \to \infty} f_n \right) \, d\mu = \int f \, d\mu
\]

Theorem (Fatou’s Lemma)

Let \((X, \mathcal{F}, \mu)\) be a measure space and \((f_n)\) be a sequence of functions in \(M^+(X, \mathcal{F})\). Then we have

\[
\int \left( \lim_{n \to \infty} f_n \right) \, d\mu \leq \lim_{n \to \infty} \int f_n \, d\mu.
\]

where \(\lim\) denotes limit infimum.

Corollary

If \((f_n)\) in theorem 7.4 above converges to \(f\) then Fatou’s Lemma gives

\[
\int f \, d\mu \leq \lim_{n \to \infty} \int f_n \, d\mu
\]

Remark

Note that in corollary 7.5 above since \(f_n\) converges to \(f\) we have that

\[
\lim f_n = f
\]

Hence

\[
\int f \, d\mu = \int \lim f_n \, d\mu \leq \lim \int f_n \, d\mu.
\]

Remark

Let \((X, \mathcal{F}, \mu)\) be a measure space and \(f : X \to \mathbb{R}_e\) be \(\mathcal{F}\)-measurable. Then we have \(f \in L\) \((X, \mathcal{F}, \mu)\) if

Furthermore,

\[
\left| \int f \, d\mu \right| \leq \int |f| \, d\mu.
\]
Theorem (Lebesgue dominated convergence theorem)

Let \((X, \mathcal{F}, \mu)\) be a measure space and \((f_n)\) be a sequence of measurable functions on \(X\) which converges \(\mu.a.e\) to \(f\). Let \(g \in L(X, \mathcal{F}, \mu)\) such that \(g\) is on \(X\).

Remark

Note that in the Monotone convergence theorem there is no analogue for a monotonic decreasing sequence of functions \((f_n)\) as the example below shows.

Readings


Compiled List of Compulsory Readings (Complete reference + abstract/rationale)

- Mathematical Analysis 1 by Elias Zakon, The Trillia Group.
Compiled List of Multimedia Resources and Useful Links

(Complete reference + abstract/rationale)

**Reading 1:** Wolfram MathWorld  (visited 03.11.06)

**Complete reference:** [http://mathworld.wolfram.com](http://mathworld.wolfram.com)

**Abstract:** Wolfram MathWorld is a specialised on-line mathematical encyclopedia.

Rationale: It provides the most detailed references to any mathematical topic. Students should start by using the search facility for the module title. This will find a major article. At any point students should search for key words that they need to understand. The entry should be studied carefully and thoroughly.

**Reading 2:** Wikipedia  (visited 03.11.06)


**Abstract:** Wikipedia is an on-line encyclopedia. It is written by its own readers. It is extremely up-to-date as entries are continually revised. Also, it has proved to be extremely accurate. The mathematics entries are very detailed.

**Rationale:** Students should use wikipedia in the same way as MathWorld. However, the entries may be shorter and a little easier to use in the first instance. They will, however, not be so detailed.

**Reading 3:** MacTutor History of Mathematics (visited 03.11.06)

**Complete reference:** [http://www-history.mcs.standrews.ac.uk/Indexes](http://www-history.mcs.standrews.ac.uk/Indexes)

**Abstract:** The MacTutor Archive is the most comprehensive history of mathematics on the internet. The resources are organised by historical characters and by historical themes.

**Rationale:** Students should search the MacTutor archive for key words in the topics they are studying (or by the module title itself). It is important to get an overview of where the mathematics being studied fits in to the history of mathematics. When the student completes the course and is teaching high school mathematics, the characters in the history of mathematics will bring the subject to life for their students. Particularly, the role of women in the history of mathematics should be studied to help students understand the difficulties women have faced while still making an important contribution. Equally, the role of the African continent should be studied to share with students in schools: notably the earliest number counting devices (e.g. the Ishango bone) and the role of Egyptian mathematics should be studied.
Module Summary

We note that having gone through this module you should now be fully equipped with the knowledge of the concepts involved in the following contents:

In unit 1 the main task has been to understand the structure of a general metric space. The grasping of concepts such as interior points, limit points or open sets and closed sets together with their combinations, is essential, properties of functions defined on metric spaces are well exposed in this unit. Leading to some of the most important results in analysis like those covering the concepts of continuity and compactness.

In unit 2 the main aspect of understanding is that of removing the geometric structure dealing with the concept of distance from a metric space and replacing it with a set based structure which constitutes a topological space. The analysis of the concepts is then carried out in a similar manner up to the level of continuity and homeomorphisms.

Finally, in units 3 we have dealt with the structure that combines both geometric and algebraic components. Thus in concepts like Lebesgue Outer Measure and Lebesgue Measure on the real line have both algebraic combination of sets like union and measurements like length of an interval. A good grasp of the properties of the Outer Measure leads to easy understanding of Lebesgue measurable subsets of the real line. A study of measurable functions on a measurable space is analogous to that of continuous functions on topological spaces. This leads to the definition of an abstract integral on an abstract measure space.

References

- Mathematical Analysis 1, Elias Zakon, 1973, The Trillia Group, Indiana, USA
- Theory of functions of a real variable, 1990, Lynn Loomis and Shlomo Stenberg, Jones and Bartlett, Boston, USA