Foreword

The African Virtual University (AVU) is proud to participate in increasing access to education in African countries through the production of quality learning materials. We are also proud to contribute to global knowledge as our Open Educational Resources are mostly accessed from outside the African continent.

This module was developed as part of a diploma and degree program in Applied Computer Science, in collaboration with 18 African partner institutions from 16 countries. A total of 156 modules were developed or translated to ensure availability in English, French and Portuguese. These modules have also been made available as open education resources (OER) on oer.avu.org.

On behalf of the African Virtual University and our patron, our partner institutions, the African Development Bank, I invite you to use this module in your institution, for your own education, to share it as widely as possible and to participate actively in the AVU communities of practice of your interest. We are committed to be on the frontline of developing and sharing Open Educational Resources.

The African Virtual University (AVU) is a Pan African Intergovernmental Organization established by charter with the mandate of significantly increasing access to quality higher education and training through the innovative use of information communication technologies. A Charter, establishing the AVU as an Intergovernmental Organization, has been signed so far by nineteen (19) African Governments - Kenya, Senegal, Mauritania, Mali, Cote d’Ivoire, Tanzania, Mozambique, Democratic Republic of Congo, Benin, Ghana, Republic of Guinea, Burkina Faso, Niger, South Sudan, Sudan, The Gambia, Guinea-Bissau, Ethiopia and Cape Verde.

The following institutions participated in the Applied Computer Science Program: (1) Université d’Abomey Calavi in Benin; (2) Université de Ouagadougou in Burkina Faso; (3) Université Lumière de Bujumbura in Burundi; (4) Université de Douala in Cameroon; (5) Université de Nouakchott in Mauritania; (6) Université Gaston Berger in Senegal; (7) Université des Sciences, des Techniques et Technologies de Bamako in Mali; (8) Ghana Institute of Management and Public Administration; (9) Kwame Nkrumah University of Science and Technology in Ghana; (10) Kenyatta University in Kenya; (11) Egerton University in Kenya; (12) Addis Ababa University in Ethiopia; (13) University of Rwanda; (14) University of Dar es Salaam in Tanzania; (15) Université Abdou Moumouni de Niamey in Niger; (16) Université Cheikh Anta Diop in Senegal; (17) Universidade Pedagógica in Mozambique; and (18) The University of the Gambia in The Gambia.

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AVU Multinational Project II funded by the African Development Bank.
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Course Overview

Welcome to Linear Algebra

The Linear Algebra is a branch of Mathematics that studies systems of linear equations and the property of matrices. It is one of the sectors with the vast and varied applications. The matrix calculus, vector calculus, linear applications and the design values and eigenvectors of an endomorphism have wide application in various branches of knowledge, particularly in the computer industry. Moreover, their concepts and developments lend themselves to multiple interpretations and the most diverse uses.

Prerequisites

- Body Concept - The Corps of Real Numbers;
- Sum - symbol and properties
- Multivariable calculus (this introduces vectors, matrices and three dimension coordinate system)

Materials

The materials required to complete this course are:

- LAY, D.C., Linear Algebra and its Application, Addison-Wesley Production Company, 1994
- https://en.wikipedia.org/wiki/Linear_algebra

Course Goals

Upon completion of this course the learner should be able to, the student must be able to:

- Identify and solve a system of linear equations
- Perform row reduction and echelon forms
- Operate with matrices including inverses
- Determine the value of the determinant of a matrix, and solving linear equations,
- Identify real vector spaces and subspaces
- Identify a linear application,
- Determine values and vectors own an endomorphism (matrix),
- Identify endomorphism (matrix) diagonalizable
Course Overview

- Linear Transformation
- Fill the MATLAB software as a tool for performing calculations along the way.

Units

Unit 0: Pre-Assessment

We study the properties of a body and analyzes the field of real numbers. The Fill the sum symbol and makes the analysis of their properties.

Unit 1: System of linear equations. Matrix and Determine

solved systems of linear equations (SEL) by applying the Gaussian elimination. Add, multiply and identify some types of matrices. Identifies invertible matrices and calculate their inverses. It is the matrix representation of SEL and hence its resolution by matrix means. Calculate determinant of a matrix.

It makes a brief introduction to MATLAB software.

Unit 2: Real Vector Space

We are introduced to the concepts of space and vector subspace on the field of real numbers. Determine the base and the dimension of vector space of finite dimensions. It introduces the concept of domestic product and is studied vector spaces of finite-dimensional inner product. Makes use of MATLAB software tools to work with matrices.

Unit 3: Linear Applications

It introduces the concept of linear transformation and identifies an isomorphism. It relates the various matrices and applying a linear system and determine the change of basis matrix. Use the MATLAB software tools to solve problems involving linear applications.

Unit 4: Diagonalization and endomorphism matrix: Quadratic form.

Introduces the concepts of vector itself associated with a specific value, diagonalizable endomorphism, similar matrices and diagonalizable matrix. Study the Deputy endomorphisms and diagonalizes up symmetric matrices. The bilinear forms is studied, highlighting the quadratic forms.

Assessment

In each unit are included formative assessment tools to check the progress of (a) s student (a) s. At the end of each module are presented summative assessment tools such as tests and term papers, which comprise the knowledge and skills studied in the module. The implementation of summative evaluation instruments is at the discretion of the institution offering the course. The suggested evaluation strategy is as follows:
## Schedule

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<thead>
<tr>
<th>Unit 1: System of linear equations, Matrix and determining</th>
<th>Activities</th>
<th>Estimated time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matrix: square matrix (triangular, diagonal, climbing and identity); transpose of a matrix and properties (symmetric and anti-symmetric matrix); summing and multiplication of matrices and properties; Multiplying a matrix by a scalar and properties; elementary operations (on line and on column); matrix in echelon form and rank of a matrix; System of linear equations: matrix representation; classification as solutions; equivalent systems and settlement systems; Regular array: regular array of criteria; inverse calculation of a regular array; Determinant of a square matrix: Laplace’s theorem on the determinant calculation; application of elementary row operations in determining the calculation; determining properties; and determining the inverse of a matrix; Cramer system. MATLAB: input and output; matrix operations; elementary operations; inverse calculation.</td>
<td></td>
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<td>25 hours</td>
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### Formative assessment

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### Summative Evaluation I

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### Summative Evaluation II

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# Course Overview

## Unit 2: Real Vector Space

| Real vector space: arbitrary vector space and properties; vector space and vector space of polynomials; linear combination of vectors; equivalent vectors systems and properties; generator space; linear dependence and independence and properties; of a matrix and linear independence of vectors rows and columns of a matrix; maximal independent subsystem and a feature vector system; Steinitz theorem and its consequences; basis and dimension of vector space; canonical basis of the vector space and the vector space of polynomials. |
|---|---|

### Vector Subspace

- Vector subspace: vector subspace of finite-dimensional space; sum, intersection, direct sum and meeting subspaces; complementary subspaces.

### Vector Space with Inner Product

- Vector space with inner product: domestic product and standard (their properties); domestic product in finite-dimensional space and matrix of the metric; orthogonal vectors; orthogonal projection; components of a vector relative to a system of orthogonal vectors; orthonormal basis and process of Gram-Schmidt orthonormalized; orthogonal complement of a vector subspace; external product in three-dimensional space (particular case of MATLAB: operations with vectors).

## Unit 3: Linear Applications

<table>
<thead>
<tr>
<th>Linear application: properties; classification (endomorphism; monomorphism, epimorphism, isomorphism and automorphism); core and void; image and characteristic; Matrix of a linear transformation: matrix calculation of a linear transformation (change of basis matrix); operations with linear applications related matrices that represent them; relations between matrices of a linear transformation; equivalent and similar matrix arrays; Deputy of a linear transformation: definition and properties; Endomorphisms deputies: properties; relationship between arrays of Assistant endomorphisms; matrix of a self-adjoint endomorphism.</th>
<th>30 hours</th>
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MATLAB: linear transformation.
Endomorphism diagonalizable and diagonalizable matrix:
- relationship between diagonalizable endomorphism and diagonalizable matrix of an endomorphism; calculation of powers, natural exponent of diagonalizable matrix.
- Eigenvector associated with a particular value in an endomorphism: Calculation of themselves and their own values vectors; subspace itself associated with a specific value; geometric and algebraic multiplicity multiplicity of self-worth; criteria endomorfismo diagonalizable;
- Diagonalization of self-adjoint endomorphism and symmetric matrix diagonalization.
- Bilinear forms: Mother bilinear form; quadratic forms.
- MATLAB: Calculation of eigenvalues and vectors; diagonalization of matrices;

**Readings and Other Resources**

The readings and other resources in this course are:

**Unit 0**

Required readings and other resources:

- [https://en.wikipedia.org/wiki/Linear_algebra](https://en.wikipedia.org/wiki/Linear_algebra)

**Unit 1**

Required readings and other resources:

- LAY, D.C., Linear Algebra and its Application, Addison-Wesley Production Company, 1994

Optional readings and other resources:

- [https://en.wikipedia.org/wiki/Linear_algebra](https://en.wikipedia.org/wiki/Linear_algebra)
Unit 2
Required readings and other resources:
LAY, D.C., Linear Algebra and its Application, Addison-Wesley Production Company, 1994
Optional readings and other resources:
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Unit 3
Required readings and other resources:
LAY, D.C., Linear Algebra and its Application, Addison-Wesley Production Company, 1994
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https://en.wikipedia.org/wiki/Linear_algebra
Unit 4
Required readings and other resources:
LAY, D.C., Linear Algebra and its Application, Addison-Wesley Production Company, 1994
Optional readings and other resources:
https://en.wikipedia.org/wiki/Linear_algebra
Unit 0. Pre-Assessment

Unit Introduction

The purpose of this unit is to determine your grasp of knowledge related to this course.

Unit Objectives

Upon completion of this unit you should be able to:

- Determine whether an operation defined in a set or is a binary operation.
- Determine if a given structure or is a body.
- Characterize the field of real numbers.
- Determine the value of an expression to sum.
- Represent a sum using the summation symbol.

Key Terms

Binary operation: X is a different set of empty set. It is said that a binary operation * is X, if for all a, b X, there has been a * b X.

Group: * is a binary operation on a set G ≠ . It is said that (G, *) is a group:

1. ∀ a, b, c ∈ G, tem-se a*(b*c) = (a*b)*c (associativity of *)
2. ∃ e ∈ G such that ∀ a ∈ G, a * e = e * a = a (existence of neutral element *)
3. ∀ a ∈ G ∃ b ∈ G b such that b = b * = a * e (every element has opposite, symmetrical or with respect to *)

If in addition, ∀ a, b ∈ G has a * b = b * a (reciprocal *), it is said that the group is commutative.

Ring: Be + and · two arbitrary binary operations on a set A ≠ ∅. It is said that (A, +, ·) is a ring that:

1. (A, +) is a commutative group
2. ∀ a, b, c ∈ A, has a·(b·c) = (a·b)·c (associativity of ·)
3. ∀ a, b, c ∈ A, it has been
4. a·(b+c) = (a·b) + (a·c) e (b+c)·a = (b·a) + (c·a) (distributivity of · compared to +).

If a ring (A, +, ·) ∀ a, b ∈ A has a · b = b · a (commutativity of ·), the ring is said to be commutative.
If a ring \((A, +, \cdot)\), \(\exists e' \in A\) such that \(\forall a \in A\) has the \(a \cdot e' = e' \cdot a = a\) (neutral existence of \(\cdot\) element), the ring is said to be unit.

In the ring \((A, +, \cdot)\) the neutral element in relation to said to \(+\) zero ring and is represented by 0.

In a unitary ring \((A, +, \cdot)\) the neutral element in relation to said to \(\cdot\) drive ring and is represented by 1.

Length: a body \((C, +, \cdot)\) is a commutative ring unit and in that any element different from \(C\) ring has zero inverse (or symmetrical) with respect to \(\cdot\), i.e. \(\forall a \in C\) and \(\neq 0\) \(\exists b \in A\) such that \(a \cdot b = b \cdot a = 1\).

The summation symbol: There is, \(a_i, i \in \mathbb{N}_0\), a sequence of real numbers. For any \(m, n \in \mathbb{N}_0\) and \(m \leq n\), the symbol \(\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n\)

The variable it is said to be the sum of the index and may be replaced by any other variable. \(m, n\) are, respectively, lower limit and upper limit of the sum.

Properties:
1. Additive property \(\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i\)
2. Homogeneous property \(\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i\)
3. Telescopic property \(\sum_{i=m}^{n} (a_k - a_{k+1}) = a_m - a_{n+1}\)
4. Sum of a constant \(\sum_{i=m}^{n} c = (n-m+1)c\)

Unit Assessment

Check your understanding!

Diagnostic test on the notion of the field and the sum

Instructions

This is a diagnostic test, for information purposes only, designed to measure your input profile in the course of Discrete Mathematics

Grading Scheme

There are ten questions, each of which is worth 20 points, totaling 200 points.

Rating Scale:
Fail: 0-100 (exclusive);
Pass: 100-140 (exclusive);
Good: 140-170 (exclusive);
Very Good: 170-200.

Students with Pass, Good and Very Good are considered to be in a position to start the module.
Feedback

1. Consider the usual operations of addition $+$ and multiplication, $\cdot$ of real numbers.

Justify that $(\mathbb{Z}, +, \cdot)$ is a commutative and unitary ring (ring of integers).

Make sure $(\mathbb{Q}, \cdot)$ is a group.

Justify that $(\mathbb{R}, +, \cdot)$ is a body (the body of the real numbers).

Say if the usual subtraction operations $-$ and division, $\div$, defined in the set of real numbers are binary operations.

2. Consider $\theta$ operation defined in the set of relative integers $\mathbb{Z}$ as follows: $\forall a, b \in \mathbb{Z}$, has $a\theta b = a + b - 3$, Where $+$ and $-$ are the usual operations of addition and subtraction, respectively.

Justify that $\theta$ is a binary operation in $\mathbb{Z}$.

Show that $(\mathbb{Z}, \theta)$ is a commutative group.

3. Calculate:

   $\sum_{i=1}^{100} 3i$.

   $\sum_{k=2}^{n}(2^k-2^{k-1})$.

   Represent the following sum using the summation symbol:

   $b_0 + b_1 x + b_2 x^2 + \cdots + b_{50} x^{50}$.

   $3^3 + 4^3 + \cdots + [10]^3$.

Unit Readings and Other Resources

The readings in this unit are to be found at the course-level section “Readings and Other Resources”.

Unit 1. System of Linear Equation: Matrix and Determinant

Unit Introduction

This unit introduces the concept of matrices and presents some matrix operations, crucial for the implementation of the Linear Algebra content.

It makes up the solving systems of linear equations via matrix condensation.

The concept of determinant of a square matrix and some of its applications is presented.

Unit Objectives

Upon completion of this unit you should be able to:

• Operate with matrices;
• Reduce matrix to echelon form;
• Compute inverse of a square invertible matrix;
• Calculate the determinant of a square matrix;
• Solve system of linear equations via matrix condensation;
• Determine the rank of a matrix

Key Terms

Real Matrix: Let m,n ∈ N. Let the sets {1,2,3,...,m} and {1,2,3,...,n}. There is an application

\[ A: \{1,2,3,...,m\} \times \{1,2,3,...,n\} \rightarrow R \]

\[ (i,j) \mapsto a_{ij} \]

that is, for all i ∈ {1,2,3,...,m}, for all j ∈ {1,2,3,...,n}, A(i,j)=a_{ij} ∈ R. The images in this application arranged in a table with rows and columns as follows:

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

This table is called matrix-type (or size) m×n, in the domain of real numbers R.
Each element in a matrix is called entry. A matrix on \( \mathbb{R} \) is said to be real matrix or matrix with entries \( \mathbb{R} \).

To locate an entry, two indices are used in this order: the row index and the column index. An element that is at the intersection of row \( i \) and column \( j \) is called entry \((i,j)\).

The set of real matrices of size \( m \times n \) is represented by \( \mathbb{R}^{(m \times n)} \).

Normally a matrix is represented by the same letter (Roman Alphabet: A, B, C, D, etc) used in the application of the origin. For example,

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

or the abbreviation, \( A = ([a_{ij}]_{(m \times n)}) \)

Echelon form of a matrix and Reduced row echelon form: a real matrix is said to be in reduced row echelon form if it satisfies the following properties:

- Zero rows, if there is any, appear at the bottom of the matrix.
- The first nonzero entry from the left of a nonzero row is a 1. This entry is called a leading one or pivot of its row.
- For each nonzero row, the leading one appears to the right and below any leading ones preceding row.
- If the column contains a leading one, then all the entries.

System of \( m \) linear equation on a field: Let \( m, n \in \mathbb{N} \), a system of \( m \) linear equation with unknown \( x_1, x_2, \ldots, x_n \) on \( \mathbb{R} \), is any combination of \( m \) linear equations, these unknowns which reduces the shape:

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots & \quad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{align*}
\]
Echelon form of a matrix is said to be reduced row echelon form if the pivot is 1 and is the only nonzero entry in the column.

Rank of a matrix: The rank of a matrix is equal to the number of nonzero rows (which is equal to the number of pivot). Given a real matrix $A$, of arbitrary dimension, the rank of $A$, denoted by $r(A)$ or $r(A)$, is equal to the reduced row echelon form matrix obtained after performing elementary row operations.

is called canonical form of the system where $a_{ij}$, $b_i \in \mathbb{R}, i = \{1, m\}$ and $j = \{1, n\}$, are, respectively, the coefficients and the terms, and for all $i \in \{1, 2, \ldots, m\}$, there exist one $j \in \{1, 2, \ldots, n\}$ such that $a_{ij} \neq 0$.

Classification of a system with respect to the solutions: A system of $m$ linear equation with $n$ unknowns on $\mathbb{R}$, is said to be possible if there is only one solution and is said to be impossible if there is none.

Solving a system of linear equation. Solving a system of $m$ linear equation in $n$ unknowns is determining the solution set (set of all solution) or conclude that it is impossible.

Learning Activities

Activity 1.1 - Matrices and Operations

Introduction

This activity introduces the concept of matrices and some matrix operations. Also presents some special types of matrices.

Activity Details

Real Matrix: Let $m,n \in \mathbb{N}$. Let the sets $\{1, 2, 3, \ldots, m\}$ and $\{1, 2, 3, \ldots, n\}$. There is an application $A: \{1, 2, 3, \ldots, m\} \times \{1, 2, 3, \ldots, n\} \rightarrow \mathbb{R}$

$\quad (i,j) \leftrightarrow a_{ij},$

that is, for all $i \in \{1, 2, 3, \ldots, m\}$, for all $j \in \{1, 2, 3, \ldots, n\}$, $A(i,j) = a_{ij} \in \mathbb{R}$ The images in this application arranged in a table with rows and columns as follows:
This table is called matrix-type (or size) \( m \times n \), in the domain of real numbers \( \mathbb{R} \).

Each element in a matrix is called entry. A matrix on \( \mathbb{R} \) is said to be real matrix or matrix with entries \( \mathbb{R} \).

To locate an entry, two indices are used in this order: the row index and the column index. An element that is at the intersection of row \( i \) and column \( j \) is called entry \((i,j)\).

The set of real matrices of size \( m \times n \) is represented by \( \mathbb{R}^{(m \times n)} \).

Normally a matrix is represented by the same letter (Roman Alphabet: A, B, C, D, etc) used in the application of the origin. For example, or the abbreviation, \( A=\begin{bmatrix} a_{ij} \end{bmatrix}_{(m \times n)} \)

Example. The real matrix 
\( A=\begin{bmatrix} a_{ij} \end{bmatrix}_{(4 \times 4)} \) such that for all \( i,j \in \{1,2,3,4\} \), we have
\( a_{ij}=\begin{cases} 1 & \text{if } |i-j|>1 \\ -1 & \text{if } |i-j|\leq1 \end{cases} \), It is of the form

\[
A = \begin{bmatrix}
  -1 & -1 & -1 & -1 \\
  -1 & -1 & -1 & -1 \\
  1 & -1 & -1 & -1 \\
  1 & 1 & -1 & -1
\end{bmatrix}
\]

Some types of matrices

Rectangular Matrix: This is a matrix in which the number of column is different from the number of row.

Square Matrix: This is a matrix in which the number of column is the same as the number of row. If these numbers are equal to \( n \in \mathbb{N} \), then it is called square matrix of order \( n \).

Row Matrix: This is a matrix of only one row.

Column Matrix: This is a matrix of only one column.

Zero Matrix: This is a matrix (square or rectangular) with all the entries equal to zero. The zero matrix of order \( m \times n \) represented by \( 0_{(m \times n)} \), and if \( m=n \), that is, if the zero matrix is rectangular, it represented by \( 0_n \).

Classification of square matrix

Let \( A=\begin{bmatrix} a_{ij} \end{bmatrix}_{(n \times n)} \) be a real square matrix of order \( n \in \mathbb{N} \).
The diagonal elements of the matrix are those which have the same row and column indexes, that is, $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$. This is called the principal diagonal or main diagonal. The sum is called the trace, that is, $\text{tr}(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$.

The real square matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{(n \times n)}$ is said to be upper triangular if the element below the main diagonal are all equal to zero, that is, for all $i \in \{1, 2, 3, \ldots, n\}$ and $i > j$, then $a_{ij} = 0$.

The real square matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{(n \times n)}$ is said to be lower triangular if the element above the main diagonal are all equal to zero, that is, for all $i \in \{1, 2, 3, \ldots, n\}$ and $i < j$, then $a_{ij} = 0$.

The real square matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{(n \times n)}$ is said to be triangular if it has lower or upper triangular

The real square matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{(n \times n)}$ is said to be diagonal for all $i \in \{1, 2, 3, \ldots, n\}$, for all $j \in \{1, 2, 3, \ldots, n\}$, if $i \neq j$, then $a_{ij} = 0$, that is, all non diagonal entries are zero.

The real square matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{(n \times n)}$ is said to be scalar if it is diagonal and there exist a real scalar $c$ such that, $a_{ii} = c$, for all $i \in \{1, 2, 3, \ldots, n\}$.

The real square matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{(n \times n)}$ is said to be identity if it is a scalar and diagonal elements are all equal to 1. The identity matrix of order $n$ represented by $I_n$.

**Matrix Operations**

Definition (Equal Matrices). Two matrices of the same dimensions are said to be equal if their corresponding elements (those having the same row and column index) are equal, that is, given the matrices $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{(m \times n)}$, $B = \begin{bmatrix} b_{ij} \end{bmatrix}_{(m \times n)}$, $A = B$ if, only if $a_{ij} = b_{ij}$, for all $i \in \{1, 2, 3, \ldots, m\}$, for all $j \in \{1, 2, 3, \ldots, n\}$.

Definition (Sum of matrices). One can add two matrices of the same dimension. In this case, given matrices $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{(m \times n)}$, $B = \begin{bmatrix} b_{ij} \end{bmatrix}_{(m \times n)} \in \mathbb{R}^{(m \times n)}$, $A + B = \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}_{(m \times n)} \in \mathbb{R}^{(m \times n)}$, for all $i \in \{1, 2, 3, \ldots, m\}$, for all $j \in \{1, 2, 3, \ldots, n\}$.

**Properties of addition of matrices**

Let the matrices be $A, B, C \in \mathbb{R}^{(m \times n)}$. Then:

1. **Commutativity**: $A + B = B + A$;
2. **Associativity**: $(A + B) + C = A + (B + C)$;
3. The addition of the neutral element in $\mathbb{R}^{(m \times n)}$ matrices is the zero matrix $0_{(m \times n)}$, that is, for any matrix $A \in \mathbb{R}^{(m \times n)}$, $A + 0_{(m \times n)} = 0_{(m \times n)} + A$;
4. For any matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{(m \times n)} \in \mathbb{R}^{(m \times n)}$, there exist one and only matrix $-A = \begin{bmatrix} -a_{ij} \end{bmatrix}_{(m \times n)} \in \mathbb{R}^{(m \times n)}$, such that $A + (-A) = 0_{(m \times n)}$.

Definition (Subtraction of matrices). One can subtract two matrices of the same dimension. In this case, given matrices $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{(m \times n)}$, $B = \begin{bmatrix} b_{ij} \end{bmatrix}_{(m \times n)} \in \mathbb{R}^{(m \times n)}$, $A - B = \begin{bmatrix} a_{ij} - b_{ij} \end{bmatrix}_{(m \times n)} \in \mathbb{R}^{(m \times n)}$, for all $i \in \{1, 2, 3, \ldots, m\}$, for all $j \in \{1, 2, 3, \ldots, n\}$.

Definition (Product of a matrix by a real scalar).
Let \( A = [\lambda_{a_{ij}}]_{(m \times n)} \in \mathbb{R}^{m \times n} \), and let a real scalar \( \lambda \). The scalar product of \( \lambda \) and the matrix \( A \) is the matrix \( \lambda A = [\lambda a_{ij}]_{(m \times n)} \in \mathbb{R}^{m \times n} \), for all \( i \in \{1, 2, 3, \ldots, m\} \), for all \( j \in \{1, 2, 3, \ldots, n\} \).

Properties of scalar product

Let \( A, B \in \mathbb{R}^{m \times n} \) be matrices and let \( \lambda, \beta \in \mathbb{R} \). then:

1. \( \lambda (A+B) = \lambda A + \lambda B \);
2. \( (\lambda + \beta)A = \lambda A + \beta A \);
3. \( (\lambda \beta)A = \lambda (\beta A) \);
4. \( 1A = A \);
5. \( \lambda A = 0_{(m \times n)} \iff \lambda = 0 \) ou \( A = 0_{(m \times n)} \)

Example. Let the real matrices be \( A = \begin{bmatrix} 1 & 2 & -1 & 0 & -3 & 4 \end{bmatrix} \) e \( B = \begin{bmatrix} -1 & 5 & 7 & -1 & 3 & -8 \end{bmatrix} \).

Determine \(-3A+2B-A\)

\[
-3A+2B-A = \begin{bmatrix} -3 & -6 & 3 & 0 & 9 & -12 \end{bmatrix} + \begin{bmatrix} -2 & 10 & 14 & -2 & 6 & -16 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 1 & 0 & 3 & -4 \end{bmatrix} = \begin{bmatrix} -6 & 2 & 18 & -2 & 18 & -32 \end{bmatrix}
\]

Multiplication of matrices

Definition (Scalar product). Let \( L = [l_1 \ l_2 \ \ldots \ l_n] \) and \( C = [c_1 \ c_2 \ \ldots \ c_n] \) be real matrices. The scalar product of \( L \) row, of dimension \( 1 \times n \), and a matrix \( C \) column, of dimension \( n \times 1 \), is the scalar

\[
l_1c_1+l_2c_2+\ldots+l_nc_n=\sum_{i=1}^n l_i c_i
\]

Example. The scalar product of the real matrices \( L = [-1 \ 3 \ 5 \ 0] \) and \( C = [-2 \ 2 \ 3 \ 1] \) is:

\[
-1 \times (-2) + 3 \times 2 + 5 \times 3 + 0 \times 1 = 2 + 6 + 15 + 0 = 23
\]

Definition (Multiplication of matrices): The multiplication \( AB \) of real matrices \( A \) and \( B \), is possible if the number of column of matrix \( A \) is equal to the number of row of matrix \( B \). In this case, it is said that the matrices \( A \) and \( B \) are consistent for the product.

Let the real matrices \( A = [a_{ij}]_{(m \times n)} \) and \( B = [b_{ij}]_{(n \times p)} \) be compatible to the product \( AB \). Then, the matrix \( AB = [p_{ij}]_{(m \times p)} \) is such that for all \( i \in \{1, 2, 3, \ldots, m\} \), and for all \( j \in \{1, 2, 3, \ldots, p\} \), we have \( p_{ij} \) is equal to the scalar product of the row \( i \) and column \( j \), that is,

\[
p_{ij} = a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}
\]

Example. Determine the product \( AB \), where \( A = [-1 \ 2 \ -2 \ 0 \ 1 \ 3] \) and \( B = [1 \ 2 \ -2 \ 0 \ 3 \ 1 \ -1 \ 2 \ -2 \ 1 \ -2] \)

\[
AB = \begin{bmatrix} -1 & 2 & -2 \\ 0 & 1 & 3 \\ -2 & 0 & 2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & -1 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1+6+4 & -2+2+2 & -2+4+0 & -6 \\ 0+3-6 & 0+1+3 & 0-1-6 & 0+2+9 \\ 9 & -2 & 4 & -2 \\ -3 & 4 & -7 & 11 \end{bmatrix} = \begin{bmatrix} 9 & -2 & 4 & -2 \\ -3 & 4 & -7 & 11 \end{bmatrix}
\]
Note. Matrix multiplication does not apply the commutative property. Moreover, the fact that the product \( AB \) exist, does not imply the existence of the product \( BA \).

However, if \( A \) and \( B \) are real square matrix of the same order, then there exist the products \( AB \) and \( BA \), and may be the same or not.

If \( AB=BA \) then it is said that \( A \) and \( B \) are interchangeable.

Example. Determine the interchangeable matrix of the matrix \( A=\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \).

If \( B \) is interchangeable with \( A \), then \( B \) is a square matrix of order two. Then \( B=\begin{bmatrix} x & y & z & w \end{bmatrix} \) is a real matrix.

\[
\begin{align*}
AB=BA & \iff [x+z & y+w & 0 & 0 ] = [x & x & z & z ] \\
\iff \begin{cases}
x+z &= x \\
y+w &= x \\
z &= 0
\end{cases}
\iff \begin{cases}
x=y-w \\
w & \in \mathbb{R}
\end{cases}
\end{align*}
\]

Therefore, \( B=\begin{bmatrix} x & x-w & 0 & w \end{bmatrix}, x,w \in \mathbb{R}. \)

Properties of Matrix Multiplication

Let \( \lambda \) be a real scalar, let \( A, B \) and \( C \) be compatible matrices and \( I \) be an identity matrix, then

1. \( IA=A \) and \( BI=B; \)
2. \( A(BC)=(AB)C; \)
3. \( A(B+C)=AB+AC; \)
4. \( (B+C)A=BA+CA. \)
5. \( \lambda(AB)=A(\lambda B)=(\lambda B)A \)

Definition (\( N_0 \) power of a square matrix \( A \) of order \( n \)).

Let \( A \) be a square matrix of order \( n \). Then

\[
\begin{align*}
A^0 &= I_n \\
A^{m+1} &= A^m A, \forall m \in N_0
\end{align*}
\]

Example. Let \( A=\begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix} \).

\[
A^3=A^2 A=\begin{bmatrix} 1 & 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -7 & 8 \end{bmatrix}
\]

Definition (Matrix as square root of a polynomial).

Let the polynomial be \( f(x)=a_n x^n+a_{n-1} x^{n-1}+\cdots+a_1 x+a_0, \) with real coefficients \( a_i, i=0,\ldots,n \), and \( x \) is an unknown. Let \( A \) be a real square matrix of order \( p \in \mathbb{N} \). Then,

\[
f(A)=a_n A^n+a_{n-1} A^{n-1}+\cdots+a_1 A+a_0 I_p .
\]

It is said that the square matrix \( A \) is a square root of the polynomial \( f(x) \) if \( f(A)=0_p. \)

Example. The real square matrix \( A=\begin{bmatrix} 1 & 7 & 0 & 4 \end{bmatrix} \) is the root of the polynomial \( f(x)=x^2-5x+4. \) In fact,

\[
f(A)=A^2-5A+4I_2=\begin{bmatrix} 1 & 35 & 0 & 16 \end{bmatrix} +\begin{bmatrix} -5 & -35 & 0 & 20 \end{bmatrix} +\begin{bmatrix} 4 & 0 & 0 & 4 \end{bmatrix} =\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} =0_2 .
\]

Definition (Transpose of a matrix). Let the real matrix be \( A=\begin{bmatrix} [a_{ij}] \end{bmatrix}_{(m \times n)}. \) The transpose of the matrix \( A, A^t, \) is the matrix of dimension \( n \times m \) whose \( (i,j) \) is \( a_{ji}. \)

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Example. Let \( B = \begin{bmatrix} -3 & -1 & 1 & 2 & 1 & -5 \end{bmatrix} \), then \( B^t = \begin{bmatrix} -3 & 2 & -1 & 1 & 1 & -5 \end{bmatrix} \).

Properties of transpose

If \( A \) and \( B \) are matrices and a real scalar \( \lambda \). Then:

1. \( [A^t]^t = A; \)
2. \( [(\lambda A)]^t = \lambda A^t; \)
3. \( [(A+B)^t] = A^t + B^t; \)
4. \( [(AB)^t] = B^t A^t. \)

Definition (Symmetric Matrix and Skew Symmetric). Let \( A = [a_{ij}]_{n \times n} \) be a square matrix of order \( n \in \mathbb{N} \). It is said that:

1. The matrix \( A \) is symmetric if \( A = A^t \), that is, for all \( i, j \in \{1, 2, 3, \ldots, n\} \), then \( a_{ij} = a_{ji}. \)
2. The matrix \( A \) is skew symmetric if \( A^t = -A \), that is, for all \( i, j \in \{1, 2, 3, \ldots, n\} \), então \( a_{ij} = -a_{ji} \).

Note. We can conclude from the definition:

1. A square matrix with real entries is symmetric if the main diagonal elements are arbitrary and opposing elements in relation to the main diagonal (entries \( (i,j) \) and \( (j,i) \)) are equal.
2. A square matrix is said to be skew symmetric if the main diagonal elements are zero and the opposing elements in relation to the main diagonal are symmetrical.

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
-1 & 0 & \sqrt{2} \\
2 & \sqrt{2} & 3
\end{bmatrix}
\]

Example. The matrix with real entries is symmetrical; and the matrix with real entries

a. \( A^t \) is a symmetric matrix;

b. \( A + B \) and \( A - B \) are symmetric matrices;

c. \( \lambda A \) is a symmetric matrix.

Conclusion

A matrix is a table with real numbers arranged in rows and columns.

We can divide matrix into two major groups: rectangular matrices and square matrices. In the group of square matrices there are subgroups, as in the case of: triangular matrices, diagonal, and scalar identity.

It can be, among other things, add and subtract matrices of the same size, to multiply a matrix by a scalar multiply two matrices (under specific conditions) and make the transpose of a matrix. The transpose operation is the cause of symmetric and skew symmetric matrices.
Assessment

1. Let the real matrix be \( A = \begin{bmatrix} -2 & -1 & 1 & 3 & 1 & -2 \end{bmatrix} \).
   
   Determine the real matrices \( AA^t \) and \( A^t A \), and prove that they are both symmetric.

   Let the real matrix be \( A \in \mathbb{R}^{m \times n} \). Prove that the real matrix \( AA^t \) and \( A^t A \) are symmetric.

2. Let the real square matrix be \( A = \begin{bmatrix} 1 & -2 & 2 & 2 & 1 & -2 & -1 & 1 & 3 \end{bmatrix} \).
   
   Determine the real square matrices \( A + A^t \) and \( A - A^t \), and show that the first one is symmetric and the second one is skew symmetric.

   Let the real square matrix be \( A \). Prove that the real square matrix \( A + A^t \) is symmetric, while the real square matrix \( A - A^t \) is skew symmetric.

3. Determine the symmetric matrix \( B \) and the skew symmetric matrix \( C \), such that \( B + C = A \).
   
   Show that the real square matrix \( B = \begin{bmatrix} -5 & 6 & -9 & 10 \end{bmatrix} \) is the root of the polynomial \( f(x) = x^2 - 5x + 4 \).

4. Find the real matrix \( A \), in the following expression:
   
   \[ 2A - \begin{bmatrix} 1 & 0 & -2 & 4 & 7 & 3 \end{bmatrix}^t = \begin{bmatrix} 2 & 0 & -3 & 4 & 0 & 8 \end{bmatrix} \, . \]

5. Consider the real matrices \( A = [a_{ij}], i \in \{1,2,3,4\}, j \in \{1,2,3\} \) and \( a_{ij} = 1/(i+j-1) \), \( B = [b_{ij}], i,j \in \{1,2,3\} \) and \( b_{ij} = i-j+1 \), \( C = [c_{ij}], i,j \in \{1,2,3\} \) and \( c_{ij} = 1/(i+j) \) .
   
   Show which of the following are defined and determine the values of the following:

   \[ AB; \]

   \[ (B-C)A; \]

   \[ A(B-C); \]

   \[ A^t C^3; \]

   \[ f(B), \text{ where } f(x) = x^3 - 3x + 1; \]

   \[ I_4 A. \]

Activity 1.2 - Reduce Row Echelon Form. System of linear equation

Introduction

In this activity, we will be doing reduced row echelon form, using Gaussian Elimination -Jordan-Gauss , and thus introduces the concept to a matrix .

Generalizes the concept of linear equations , and presents a new method pair to its resolution based on the condensation of matrices.

Use is made of a matrix to discuss a system of linear equations.
Activity Details

Elementary Row Operations (column) on a Matrix

Definition (Elementary Operations). An elementary row (column) operation on a matrix $A$ is any one of the following operations:

Type 1: Interchange any two rows

Type 2: Multiply a row (column) by a nonzero number

Type 3: Add a multiple of one row (column) to another

Representation of elementary operations:

- **Elementary Row Operations**
  - Interchanging rows $L_i$ and $L_k$, is represented by $L_i \leftrightarrow L_k$;
  - Multiplying the row $L_i$ by a nonzero real scalar $\lambda$, represented by $L_i \leftarrow \lambda L_i$
  - Replacing the row $L_i$ by the row $L_i + \lambda L_k$, represented by $L_i \leftarrow L_i + \lambda L_k$.

- **Elementary Column Operations**
  - Interchanging columns $C_j$ and $C_l$, represented by $C_j \leftrightarrow C_l$;
  - Multiplying the column $C_j$ by a nonzero real scalar $\beta$, represented by $C_j \leftarrow \beta C_j$;
  - Replacing the column $C_j$ by the column $C_j + \lambda C_l$, represented by $C_j \leftarrow C_j + \lambda C_l$.

Exemple. Let the real matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 0 & -1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$$

1. Interchanging rows 1 and 3 of $A$, we obtain

$$A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 0 & -1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 0 & -1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}_{L_1 \leftrightarrow L_3} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & -1 & 3 \\ 2 & 0 & 0 & 4 \end{bmatrix}$$

2. Multiplying row 1 of $A$ by $1/2$

$$A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 0 & -1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 0 & -1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}_{L_1 \leftarrow \frac{1}{2} L_1} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & -1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$$
3. Replacing row 3 by the sum of row 3 plus row 2 multiply by 4:

\[ A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 0 & -1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 0 & -1 & 3 \\ 0 & -2 & 1 & -3 \end{bmatrix} \]

4. Interchange column 1 and 3:

\[ A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 0 & -1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 2 & 4 \\ -1 & 0 & 1 & 3 \\ 0 & -2 & 1 & 0 \end{bmatrix} \]

5. Multiplying column 4 by -3:

\[ A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 0 & -1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & -12 \\ 1 & 0 & -1 & -9 \\ 1 & -2 & 0 & 0 \end{bmatrix} \]

6. Replace column 2 by the sum column 2 plus column 4 multiply by -2

\[ A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 0 & -1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -8 & 0 & 4 \\ 1 & -6 & -1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} \]

Echelon form and Rank of a Matrix

Definition (Echelon form of a Matrix and reduced row echelon form). It is said that a real matrix is in echelon form if:

- All zero row, if there is any, appear at the bottom of the matrix.
- The first nonzero entry of each row - pivot - is located in a column far left that all pivot of the following rows.
- A echelon matrix is said to be in reduced echelon form if the pivot is equal to 1 and is the unique entry different from zero in that column.

Example. These real matrices

\[ A = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad e \quad B = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix} \]

are in echelon form and the matrix A is in reduced echelon form.

Example. Any nonzero matrix is in reduced echelon form. Any row matrix is in echelon form, and it is in reduced echelon form if the pivot is equal 1.
Reduction of a matrix in echelon form or reduced echelon form

Given a real matrix A of dimension m×n, it is possible to obtain from A an echelon form of a matrix or stepwise reduced by a sequence of elementary row operation. This process is called reduction of a matrix in echelon form.

The method used to carry out row reduction is Gaussian elimination. Gaussian elimination consists of two steps:

Step 1: The transformation of the augmented matrix [A|b] to the matrix [C|d] in row echelon form using elementary row operation

Step 2: Solution of the linear system corresponding to the augmented matrix [C|d] using back substitution

Example. Let the real matrix

\[
A = \begin{bmatrix}
0 & 3 & -1 \\
2 & 1 & 2 \\
1 & 2 & 1 \\
3 & -1 & 0
\end{bmatrix}.
\]

The matrix reduction to reduced echelon form

\[
\begin{align*}
\begin{bmatrix}
0 & 3 & -1 \\
2 & 1 & 2 \\
1 & 2 & 1 \\
3 & -1 & 0
\end{bmatrix} & \rightarrow \begin{bmatrix}
1 & 2 & 1 \\
0 & 3 & -1 \\
0 & 0 & 1
\end{bmatrix} & \rightarrow \begin{bmatrix}
1 & 2 & 1 \\
0 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix} & \rightarrow \begin{bmatrix}
1 & 2 & 1 \\
0 & 3 & 0 \\
0 & 0 & 1
\end{bmatrix} & \rightarrow \begin{bmatrix}
1 & 2 & 1 \\
0 & 3 & 0 \\
0 & 0 & 1
\end{bmatrix} & \rightarrow \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\end{align*}
\]

Note. Depending on the elementary row operation, a matrix can be reduced to different matrices in echelon form. However, each matrix is reduce by step regardless of elementary row operation.

Definition (Rank of a matrix). The rank of a matrix in echelon form is equal to the number of nonzero row (which is equal to the number of pivots). Given a real matrix of arbitrary dimension, the rank A which is denoted by c(A) or r(A), is equal to the number of pivots in an echelon form of A.

Example. The rank of the matrix A in the previous example is 3.
Properties of Rank of Matrix

Let the real matrices be $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, and let $\lambda$ be a nonzero real scalar. Then

$$c(A) \leq m \quad c(A) \leq n;$$
$$c(\lambda A) = c(A);$$
$$c(AB) \leq c(A) \quad c(AB) \leq c(B);$$
$$c(A^t) = c(A).$$

System of Linear Equations

Definition (Linear Equation on a field $\mathbb{R}$). Let $n \in \mathbb{N}$. It is called linear equation of unknown quantities $x_1, x_2, \ldots, x_n$ on the field $\mathbb{R}$. All equation can be reduce to the form:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

where $a_i, b \in \mathbb{R}, i = (1, n)$ and there exist at least one $i \in \{1, 2, \ldots, n\}$ such that $a_i \neq 0$.

For all $i \in \{1, 2, \ldots, n\}$, $a_i$ is the unknown coefficient of $x_i$, and $b$ is an independent term.

Definition (Solution of a Linear Equation). Solution of a linear equation with $n$ unknown $(x_1, x_2, \ldots, x_n)$, on $\mathbb{R}$,

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

is an element $s = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$, that will transform this equation true, true, $a_1 s_1 + a_2 s_2 + \cdots + a_n s_n = b$ is a true statement.

Example. The element $s = (-4, 1) \in \mathbb{R}^2$ is a solution to the linear equation $2x_1 + 3x_2 = -5$ (or $2x + 3y = -5$), because $2(-4) + 3.1 = -5$ is a true statement.

Note. It is noted that:

$$2x_1 + 3x_2 = -5 \iff x_1 = \frac{-5}{2} - \frac{3}{2} x_2 \quad \text{and} \quad x_2 = \lambda \in \mathbb{R},$$

the general solution to this equation is

$$S = \{((-5)/2 - 3/2 \lambda, \lambda) : \lambda \in \mathbb{R}\},$$

that is it has infinite solution.

Definition (System of $m$ linear equation on a field). Let $m, n \in \mathbb{N}$. A system of $m$ linear equation with unknown $x_1, x_2, \ldots, x_n$, on the field $\mathbb{R}$, is any combination of $m$ linear equation, with unknowns which are reduced to form

$$\begin{cases}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
    \vdots \quad \vdots \quad \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{cases},$$

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canonical form of the system where $a_{ij}, b_i \in \mathbb{R}, i=1(1,m)$ are the coefficient and the constant term respectively, and for all $i \in \{1,2,\ldots,m\}$, there exist $j \in \{1,2,\ldots,n\}$ such that $a_{ij} \neq 0$.

Matrices associated with the system of linear equations in canonical form (The above definition)

Coefficient of Matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix};$$

Column Matrix of Unknowns:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix};$$

Column Matrix of Constant Term:

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix};$$

Augmented Matrix:

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}. $$

Matrix Representation of Linear equation system in canonical form (The above definition):

$$AX = B.$$  

Example. The matrix representation of linear equation,

$$\begin{cases} x_1 - 2x_2 + 3x_3 + x_4 = -3 \\ 2x_1 - x_2 + 3x_3 - x_4 = 0 \end{cases},$$

$$AX = B$$

$$\Leftrightarrow \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$
The augmented matrix is
\[
\begin{bmatrix}
A|B
\end{bmatrix} = \begin{bmatrix}
1 & -2 & 3 & 1 & -3 \\
2 & -1 & 3 & -1 & 0
\end{bmatrix}
\]

Definition (Solution of a system of linear equation). Solution of a system of m linear equation in n unknowns on R, is an element
\[
s=(s_1,s_2,\ldots,s_n) \in \mathbb{R}^n,
\]
which is a solution of each of the m linear equation.

Example. The element \(s=(1,4,1,1)\in \mathbb{R}^4\) is the solution to the system
\[
\begin{align*}
x_1 - 2x_2 + 3x_3 + x_4 &= -3 \\
2x_1 - x_2 + 3x_3 - x_4 &= 0
\end{align*}
\]
because:
\[
\begin{align*}
1 - 2 \cdot 4 + 3 \cdot 1 + 1 &= -3 \\
2 \cdot 1 - 4 + 3 \cdot 1 - 1 &= 0
\end{align*}
\Rightarrow \begin{align*}
1 - 8 + 3 + 1 &= -3 \\
2 - 4 + 3 - 1 &= 0
\end{align*}
\Rightarrow \begin{align*}
-3 = -3 \text{ (AV)} \\
0 = 0 \text{ (AV)}
\end{align*}
\]

However, the element \(s=(-7,0,1,1)\in \mathbb{R}^4\) is not a solution to this system because:
\[
\begin{align*}
-7 - 2 \cdot 0 + 3 \cdot 1 + 1 &= -3 \\
2 \cdot (-7) - 0 + 3 \cdot 1 - 1 &= 0
\end{align*}
\Rightarrow \begin{align*}
-7 - 0 + 3 + 1 &= -3 \\
-14 - 0 + 3 - 1 &= 0
\end{align*}
\Rightarrow \begin{align*}
-3 = -3 \text{ (AV)} \\
-12 = 0 \text{ (AV)}
\end{align*}
\]
In this case, \(s=(-7,0,1,1)\in \mathbb{R}^4\) is a solution to the first equation but is not a solution to the second, then this element is not a solution to the given linear equation.

Definition (Classification of a system with respect to the solutions).: A system of m linear equation with n unknowns on R, is said to be possible if there is only one solution and is said to be impossible if there is none.

Theorem (Classification of a system with respect to the solutions). Let a system of m linear equation in n unknowns on R, and a real matrix A and [A|B], the coefficient and augmented matrix respectively. Then:

a. The system is possible to determine if and only if \(c(A)=c([A|B])=n\);

b. The system is undefined if and only if \(c(A)=c([A|B])<n\);

c. The system is impossible if \(c(A)<c([A|B])\).

Definition (Degree of uncertainty of a system). Let a system of m linear equation in n unknowns on R, be possible. The number \(g=n-c(A)\) is called the degree of uncertainty of the system.

Note. The degree of uncertainty of a system gives the possible number of free variables of a system of linear equation, that is, the variables (or unknowns) can assume any real values in the overall system solution.
If the degree of uncertainty is zero, the system can be determined, or else the system will be indetermined.

The free variable (or unknowns) of a possible linear equation system is said to be dependent variables (may take a fixed value or may be a function of one or more free variables).

Example. The system

\[
\begin{align*}
x_1 - 2x_2 + 3x_3 + x_4 &= -3 \\
2x_1 - x_2 + 3x_3 - x_4 &= 0
\end{align*}
\]

is undetermined:

\[
[A|B] = \begin{bmatrix}
1 & -2 & 3 & 1 & -3 \\
2 & -1 & 3 & -1 & 0
\end{bmatrix} \longrightarrow \begin{bmatrix}
1 & -2 & 3 & 1 & -3 \\
0 & 3 & -3 & -3 & 6
\end{bmatrix}.
\]

Note that \(c(A) = c([A|B]) = 2 < 4\) (# of unknowns). The system allows two free variables \(g = 4 - 2 = 2\).

Example. The system

\[
\begin{align*}
x_1 - 2x_2 - x_3 &= 1 \\
2x_1 - 4x_2 + x_3 &= -1 \\
-x_1 + 3x_2 + 2x_3 &= 2
\end{align*}
\]

can be determined as:

\[
[A|B] = \begin{bmatrix}
1 & -2 & -1 & 1 \\
2 & -4 & 1 & -1 \\
-1 & 3 & 2 & 2
\end{bmatrix} \longrightarrow \begin{bmatrix}
1 & -2 & -1 & 1 \\
0 & 0 & 3 & -3 \\
-1 & 3 & 2 & 2
\end{bmatrix} \longrightarrow \begin{bmatrix}
1 & -2 & -1 & 1 \\
0 & 0 & 3 & -3 \\
0 & 1 & 1 & 3
\end{bmatrix} \longrightarrow \begin{bmatrix}
1 & -2 & -1 & 1 \\
0 & 0 & 3 & -3 \\
0 & 1 & 1 & 3
\end{bmatrix} \longrightarrow \begin{bmatrix}
1 & -2 & -1 & 1 \\
0 & 0 & 3 & -2 \\
0 & 0 & 3 & 1
\end{bmatrix}
\]

Note that \(c(A) = c([A|B]) = 3\) (# of unknowns).

Example. The system

\[
\begin{align*}
x_1 - 2x_2 - x_3 &= 1 \\
2x_1 - 4x_2 + x_3 &= 0 \\
2x_1 - 4x_2 - 2x_3 &= 3
\end{align*}
\]

is impossible, because:

\[
[A|B] = \begin{bmatrix}
1 & -2 & -1 & 1 \\
2 & -4 & 1 & 0 \\
2 & -4 & -2 & 3
\end{bmatrix} \longrightarrow \begin{bmatrix}
1 & -2 & -1 & 1 \\
0 & 0 & 3 & -2 \\
2 & -4 & -2 & 3
\end{bmatrix} \longrightarrow \begin{bmatrix}
1 & -2 & -1 & 1 \\
0 & 0 & 3 & -2 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
Note that \(c(A)=2<3=c([A|B])\).

Solution of system of linear equation

Definition (Solution of system of linear equation). Solve a system of \(m\) linear equation with \(n\) is to determine the solution set (set of all solutions) or conclude is impossible.

Solution of a system of \(m\) linear equation with \(n\) unknowns whose augmented matrix is echelon (not reduced):

- If \(c(A)<c([A|B])\) the system is impossible.
- If \(C(A)=c([A|B])=n\), determine the unique solution of the system, determine the value of the unknowns (or variables), and make the substitutions respectively
- If \(C(A)=c([A|B])<n\), determine the set of solution \(S\), assign parameters to the free variables (or independent), and then determine the value or algebraic expression of the dependent variables and make their replacement.

The dependent variables are associated with the pivot of the system of augmented matrix.

Example. The system

\[
\begin{align*}
x_1 - 2x_2 + 3x_3 + x_4 &= -3 \\
x_3 - x_4 &= 1 \\
2x_4 &= 4
\end{align*}
\]

is associated to the augmented matrix

\[
[A|B] = \begin{bmatrix}
1 & -2 & 3 & 1 & 3 \\
0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 2 & 4
\end{bmatrix},
\]

which is in echelon form. Note that this system is undetermined with unlimited degree of uncertainty \(g=1\) (the free variable is \(x_2\) because it is not associated with any pivot). Then:

\[
\begin{align*}
x_1 - 2x_2 + 3x_3 + x_4 &= -3 \\
x_3 - x_4 &= 1 \\
2x_4 &= 4
\end{align*} \iff \begin{align*}
x_1 - 2x_2 + 3x_3 + 2 &= -3 \\
x_3 - 2 &= 1 \\
x_4 &= 2
\end{align*} \iff \begin{align*}
x_1 - 2x_2 + 9 &= -5 \\
x_3 &= 3 \\
x_1 &= 2
\end{align*}
\]

\[
\begin{align*}
x_1 &= -14 + 2\lambda \\
x_2 &= \lambda \in \mathbb{R} \\
x_3 &= 3 \\
x_4 &= 2
\end{align*}
\]

The set of solution is

\[S = \{(-14 + 2\lambda, \lambda, 3, 2) : \lambda \in \mathbb{R}\}.\]
Example. The system

\[
\begin{align*}
x_1 - 2x_2 - x_3 &= 1 \\
x_2 + x_3 &= 3 \\
3x_3 &= -3
\end{align*}
\]

is associated with the augmented matrix

\[
[A|B] = \begin{bmatrix}
1 & -2 & -1 & 1 \\
0 & 1 & 1 & 3 \\
0 & 0 & 3 & -3
\end{bmatrix},
\]

which is in echelon form. Note that the system can be given as \( c(A) = c([A|B]) = 3 \) (# of unknowns).

Then:

\[
\begin{align*}
x_1 - 2x_2 - x_3 &= 1 \\
x_2 + x_3 &= 3 \\
3x_3 &= -3
\end{align*} \Rightarrow
\begin{align*}
x_1 - 2x_2 + 1 &= 1 \\
x_2 - 1 &= 3 \\
x_3 &= -1
\end{align*} \Rightarrow
\begin{align*}
x_1 - 8 &= 0 \\
x_2 &= 4 \\
x_3 &= -1
\end{align*} \Rightarrow
\begin{align*}
x_1 &= 8 \\
x_2 &= 4 \\
x_3 &= -1
\end{align*}

The set of solution is

\[
S = \{(8, 4, -1)\}
\]

Solution of a system of \( m \) linear equation with \( n \) unknowns whose augmented matrix is row reduced.

When the augmented matrix associated with a row reduce system, the system is solved using the same method applied when the system is not row reduced. It depends only on the “substitution”

Example. The system

\[
\begin{align*}
x_1 - 2x_2 &= -14 \\
x_3 &= 3 \\
x_4 &= 2
\end{align*}
\]

is associated with the augmented matrix

\[
[A|B] = \begin{bmatrix}
1 & -2 & 0 & 0 & -14 \\
0 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 1 & 2
\end{bmatrix}.
\]
which is row reduced. Note that this system can be finite with a degree of uncertainty $g = 4 - 3 = 1$

Then:

\[
\begin{align*}
x_1 - 2x_2 &= -14 \\
x_3 &= 3 \\
x_4 &= 2
\end{align*}
\]

\[
\begin{align*}
x_1 &= -14 + 2\lambda \\
x_2 &= \lambda \in \mathbb{R} \\
x_3 &= 3 \\
x_4 &= 2
\end{align*}
\]

The solution set is

\[S = \{(-14 + 2\lambda, \lambda, 3, 2) : \lambda \in \mathbb{R}\}\]

Definition (Equivalent Systems). Two system of linear equation with $n$ unknowns are said to be equivalent if they have the same solution set.

Theorem (Principle of equivalence of system of linear equations). We will obtain a system of linear equation equivalent to other data, if we apply the following properties:

1. Interchange the order of the equation;
2. Multiply a given equation by a nonzero scalar $\lambda$;
3. Substitute a certain equation by its sum with another equation.

Note. The equivalence principle applied to a system of linear equation, produces similar effects on the rows of its augmented matrix, that is:

Interchange between the $i$-th equation and $k$-th equation, we obtain $L_i \leftrightarrow L_k$ operation on the row of its augmented matrix;

b. Multiplying the $i$-th equation by nonzero real scalar $\lambda$, we obtain $L_i \leftarrow \lambda L_i$ in the row of its augmented matrix;

c. Substitute the $i$-th equation for its sum with the $k$-th equation, multiply by a real scalar $\lambda$, we obtain $L_i \leftarrow L_i + \lambda L_k$, on the row of its augmented matrix.

Thus, the echelon matrix (reduced or not) obtained from the augmented matrix of a given system of linear equation by applying elementary row operations, is associated with a system of linear equation associated to that system.

Then, a practical rule of solving a system a system of linear equations, is reducing the augmented matrix to echelon form (reduced or not) and from there solve the associated system.
Example. Solve the following system of linear equation:

\[
\begin{align*}
&x_1 + x_2 - x_3 = -2 \\
&x_1 - 2x_2 + x_3 = 5 \\
&-x_1 + 2x_2 + x_3 = 3 \\
\end{align*}
\]

\[
[A|B] = \begin{bmatrix}
1 & 1 & -1 & -2 \\
1 & -2 & 1 & 5 \\
-1 & 2 & 1 & 3 \\
\end{bmatrix} \xrightarrow{L_2 \rightarrow -L_1} \begin{bmatrix}
1 & 1 & -1 & -2 \\
0 & -3 & 2 & 7 \\
-1 & 2 & 1 & 3 \\
\end{bmatrix} \xrightarrow{L_3 \rightarrow L_2+L_3} \begin{bmatrix}
1 & 1 & -1 & -2 \\
0 & -3 & 2 & 7 \\
0 & 3 & 0 & 1 \\
\end{bmatrix}
\]

\[
\xrightarrow{L_3 \rightarrow -\frac{1}{3}L_3} \begin{bmatrix}
1 & 1 & -1 & -2 \\
0 & 1 & -2/3 & -7/3 \\
0 & 3 & 0 & 1 \\
\end{bmatrix} \xrightarrow{L_1 \rightarrow L_1-\frac{1}{3}L_2} \begin{bmatrix}
1 & 0 & -1/3 & 1/3 \\
0 & 1 & -2/3 & -7/3 \\
0 & 3 & 0 & 1 \\
\end{bmatrix}
\]

\[
\xrightarrow{L_3 \rightarrow L_3-3L_1} \begin{bmatrix}
1 & 0 & -1/3 & 1/3 \\
0 & 1 & -2/3 & -7/3 \\
0 & 2 & 8 & 4 \\
\end{bmatrix} \xrightarrow{L_4 \rightarrow L_4-L_3} \begin{bmatrix}
1 & 0 & -1/3 & 1/3 \\
0 & 1 & -2/3 & -7/3 \\
0 & 0 & 1 & 4 \\
\end{bmatrix}
\]

\[
\xrightarrow{L_3 \rightarrow L_3-L_4} \begin{bmatrix}
1 & 0 & 0 & 5/3 \\
0 & 1 & -2/3 & -7/3 \\
0 & 1 & 4 & 1 \\
\end{bmatrix} \xrightarrow{L_2 \rightarrow L_2-\frac{3}{5}L_3} \begin{bmatrix}
1 & 0 & 0 & 5/3 \\
0 & 1 & 0 & 1/3 \\
0 & 0 & 1 & 4 \\
\end{bmatrix}
\]

\[
\begin{align*}
&x_1 + x_2 - x_3 = -2 \\
&x_1 - 2x_2 + x_3 = 5 \\
&-x_1 + 2x_2 + x_3 = 3 \\
\end{align*} \Leftrightarrow \begin{align*}
x_1 &= \frac{5}{3} \\
x_2 &= \frac{1}{3} \\
x_3 &= 4 \\
\end{align*}
\]

Then the set of solution of this system is

\[
S = \left\{ \left( \frac{5}{3}, \frac{1}{3}, 4 \right) \right\}
\]

Example. Solve the following system

\[
\begin{align*}
x_1 - 2x_2 + 3x_3 + x_4 &= -3 \\
2x_1 - x_2 + 3x_3 - x_4 &= 0 \\
\end{align*}
\]

\[
[A|B] = \begin{bmatrix}
1 & -2 & 3 & 1 & -3 \\
2 & -1 & 3 & -1 & 0 \\
\end{bmatrix} \xrightarrow{L_1 \rightarrow L_2-2L_1} \begin{bmatrix}
1 & -2 & 3 & 1 & -3 \\
0 & 3 & -3 & -3 & 6 \\
\end{bmatrix}
\]

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then the set of solution of this system is

\[ S = \{ (1 - \lambda + \beta, 2 + \lambda + \beta, \lambda, \beta) : \lambda, \beta \in \mathbb{R} \} \]

Homogeneous System

Definition (Homogeneous System). A linear system of equation is said to be homogeneous if all the constant terms are zero.

Example. The system is homogeneous

\[
\begin{align*}
  x_1 - 2x_2 + 3x_3 + x_4 &= 0 \\
  x_2 + x_3 - 2x_4 &= 0 \\
  x_1 - x_4 &= 0
\end{align*}
\]

Note. A homogeneous system is always defined.

Definition (Homogenous of associated System). Let a system of \( m \) linear equations be \( AX = B \). The associated homogeneous system is \( AX = 0_{(m \times 1)} \). Exemple. The homogeneous system associated with the system

\[
\begin{align*}
  x_1 - 2x_2 + 3x_3 + x_4 &= -3 \\
  2x_1 - x_2 + 3x_3 - x_4 &= 0
\end{align*}
\]

Note. Note that:

a. If \( AX = B \) is fixed then the associate homogeneous is fix;

b. If \( AX = B \) is finite then the associate homogeneous is finite;

c. If \( AX = B \) is impossible then the associate homogeneous is possible (fix or finite).
Theorem. Let \( s_p \) be a particular solution of the system of \( m \) linear equation \( AX=B \). then, \( s_0 \) is a solution to this system if and only if there exist a particular solution \( s_h \) of the associate homogeneous system, \( AX=0_{(m\times 1)} \), such that \( s_0=s_p+s_h \).

Note. It follows from the previous theorem that the general solution of a system of \( m \) linear equation, \( AX=B \), is obtained by adding its own particular solution. The general solution of its associate homogeneous is \( AX=0_{(m\times 1)} \).

Also the general solution of the homogeneous associated system with \( AX=B \), is obtained by subtracting the overall solution of \( AX=B \) is a particular solution.

Example.

The general solution of the system

\[
\begin{align*}
    x_1 - 2x_2 + 3x_3 + x_4 &= -3 \\
    2x_1 - x_2 + 3x_3 - x_4 &= 0
\end{align*}
\]

is

\[ s = (1 - \lambda + \beta, 2 + \lambda + \beta, \lambda, \beta) : \lambda, \beta \in \mathbb{R}. \]

For a particular solution \( s_p \), this system can be obtain by giving concrete values to the free variables \( \lambda \) and \( \beta \). For \( \lambda=\beta=0 \), we have \( s_p=(1,2,0,0) \).

Then the general solution of the homogeneous associated system is

\[
\begin{align*}
    x_1 - 2x_2 + 3x_3 + x_4 &= 0 \\
    2x_1 - x_2 + 3x_3 - x_4 &= 0
\end{align*}
\]

\[ s - s_p = (-\lambda + \beta, \lambda + \beta, \lambda, \beta) : \lambda, \beta \in \mathbb{R}. \]

Conclusion

Depending on the basic operations of applied lines, a matrix can be reduced to different matrices in echelon form. However, each matrix reduced by stepwise regardless of elementary row operations applied on the rows.

Any real matrix has one, and only one rank.

Solving a system of linear equations with unknowns \( n \) is an element of \( \mathbb{R}^n \).

The possibility of a system of linear equations depends on the rank of its augmented matrix and a coefficient matrix.
(1) Determine the rank of the following real matrices:

\[ A = \begin{bmatrix} 1 & -1 & 2 & 1 & 2 & 1 \\ -1 & 1 & -2 & 2 & 0 & 2 \\ 1 & -2 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 1 & 1 & 2 & 1 \\ -1 & -2 & 2 & 1 & -1 & 0 \\ 1 & 1 & -1 & 2 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \]

(2) Discuss the rank of each of the following real matrices depending on the actual parameter of \((a, \alpha, \beta)\):

\[ A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ a & 1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & \alpha & 1 \\ 1 & 1 & \alpha & \alpha/2 \\ 1 & \beta & \alpha & 1 \end{bmatrix}, \quad eC = \begin{bmatrix} 2 & 1 & \alpha & 1 \\ 1 & 1 & \alpha & \alpha/2 \\ 1 & \beta & \alpha & 1 \end{bmatrix}. \]

(3) Consider the real matrix

\[ A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a & -2 \end{bmatrix}, \]

where \(a\) is a real parameter.

a) Determine the value of \(a\), such that \(c(A)<3\).

b) Determine the value of \(a\), such that, \(c(A)=3\).

c) For \(a=5\), reduce the real matrix \(A\) to echelon form, use elementary row and/or column operations

4. Discuss the following system based on the actual parameter of \(a\) and \(b\):

\[ \begin{align*}
2x + y &= b \\
3x + 2y + z &= 0 \\
x + ay + z &= 2
\end{align*} \]

(4) Solve the following system

\[ \begin{align*}
x_1 + x_2 + x_3 &= 2 \\
2x_1 + x_2 &= 3 \\
x_1 - x_2 - 3x_3 &= 0
\end{align*} \]

(a) Indicate the solution set of its associated homogeneous system.
Activity 1.3 - Determinant and its applications: Regular Matrix

Introduction

The aim of this activity is to introduce the concept of a determinant of a square matrix, and determining from the properties provided efficient and effective methods for determining the value of the determinant of a square matrix.

It introduces the concept of a regular matrix, and determine the inverse of a regular matrix by means of linear equations system and determining application.

It introduces the concept of Cramer system, and presents the Cramer method (based on the determinant calculation) for solution of such a system.

Activity Details

Definition (Permutation of natural number). Let \( S = \{1, 2, 3, \ldots, n\} \) be the set of integers from 1 to \( n \), arranged in ascending order. An rearrangement \( j_1, j_2, \ldots, j_n \) of the elements of \( S \) is called a permutation of \( S \). We can consider a permutation of \( S \) to be a one-to-one mapping of \( S \) onto itself.

Example. Let the natural numbers be 1, 2, 3. There are six permutation of these number:

\[
I = 1, 2, 3 \quad \gamma = 2, 1, 3 \quad \varepsilon = 3, 1, 2 \\
\alpha = 1, 3, 2 \quad \delta = 2, 3, 1 \quad \beta = 3, 2, 1.
\]

Note (Number of permutation). Given the natural number 1,2,\ldots,n, there exist

\[
n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1
\]

permutations of these real numbers. The set of all permutation of \( n \) natural numbers is represented by \( S_n \).

Definition (Inversion). Let \( i_1, i_2, \ldots, i_n \) be an arbitrary permutation of the natural numbers 1,2,\ldots,n. It is said that the elements \( i_j \) and \( i_k \) form an inverse if \( j < k \) and \( i_j > i_k \), that is, if \( i_j \) and \( i_k \) permutation appear in descending order.

Example. In the permutation \( \delta = 2, 3, 1 \), the natural numbers 1,2,3, 2 and 1 form an inverse 3 and 1, too.

Definition (Parity of a permutation). A permutation \( i_1, i_2, \ldots, i_n \), of the natural numbers 1,2,\ldots,n, is said even [odd] when the total number of inversion is is even [odd].

Example. The permutation \( \delta = 2, 3, 1 \), of natural numbers 1,2,3, is even (has two inversions).

\[
|A| = \sum_{j_1, j_2, \ldots, j_n} \prod_{\xi \in S_n} (-1)^{\xi} a_{1j_1}a_{2j_2}a_{3j_3} \ldots a_{nj_n}.
\]

Determinant of a square matrix. Let \( A = [a_{ij}] \) be a square matrix of order \( n \). is called a determinant of \( A \), and is represented by \( |A| \).


Example. Determine whether the determinant of the square matrix is of order 1, of order 2 and of order 3.

1. Let $A = [a_{11}]$, be a real square matrix of order 1. then, $|A| = a_{11}$.

2. Let $A = [a_{11} \ a_{12} \ a_{21} \ a_{22}]$ be a real square matrix of order 2. Now,

$$S_2 = \{ I = 1, 2; \alpha = 2, 1 \} ,$$

where $I$ is a permutation, and $\alpha$ is an odd permutation. Then, by definition,

$$|A| = \sum_{j_1, j_2 \in S_1} (-1)^{\alpha} a_{1j_1}a_{2j_2}$$

$$= (-1)^{0} a_{11}a_{22} + (-1)^{1} a_{12}a_{21}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

(1) Let $A = [a_{11} \ a_{12} \ a_{13} \ a_{21} \ a_{22} \ a_{23} \ a_{31} \ a_{32} \ a_{33}]$, be a real square matrix of order 3. Now,

$$S_3 = \{ I = 1, 2, 3; \alpha = 1, 3, 2; \gamma = 2, 1, 3; \delta = 2, 3, 1; \varepsilon = 3, 1, 2; \beta = 3, 2, 1 \} ,$$

$$|A| = \sum_{j_1, j_2, j_3 \in S_3} (-1)^{\alpha} a_{1j_1}a_{2j_2}a_{3j_3}$$

$$= (-1)^{0} a_{11}a_{22}a_{33} + (-1)^{1} a_{11}a_{23}a_{32} + (-1)^{1} a_{12}a_{21}a_{33} + (-1)^{0} a_{12}a_{23}a_{31} +$$

$$+ (-1)^{0} a_{13}a_{21}a_{32} + (-1)^{1} a_{13}a_{22}a_{31}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} -$$

$$- a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{22}a_{33} - a_{13}a_{22}a_{31}$$

The determinant of the three columns on the left is the sum of the products along the solid diagonals minus the sum of the products along the dashed diagonals.
Definition (Cofactor of an element). Let \( A = [a_{ij}] \) be a square matrix of order \( n \). is called cofactor (or algebraic complement) of an element \( a_{ij} \), the matrix \( A \), to \( n \):

\[
A_{ij} = (-1)^{i+j} |A(i|j)|,
\]

where \( A(i|j) \) is the submatrix of \( A \), which is obtained by eliminating the row \( i \) and the column \( j \) of the matrix \( A \).

Example. Let a real square matrix of order 3,

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
0 & 2 & 3 \\
-1 & 1 & 2
\end{bmatrix}.
\]

Then:

\[
A_{21} = (-1)^{2+1} |A(2|1)| = -\begin{vmatrix}
1 & 2 \\
-1 & 2
\end{vmatrix} = -(-2 - 2) = 4
\]

Theorem (Laplace Expansion). Let \( A = [a_{ij}] \) be a square matrix of order \( n \). then:

1. For all \( c \in \{1,2,3,\ldots,n\} \), we have

\[
|A| = \sum_{i=1}^{n} a_{ic}A_{ic},
\]

known method Laplacean developing along the column \( c \).

2. For all \( l \in \{1,2,3,\ldots,n\} \), we have

\[
|A| = \sum_{j=1}^{n} a_{lj}A_{lj},
\]

known method Laplacean developing along the row \( l \).
Unit 1. System of Linear Equation: Matrix and Determinant

Exemple. Let the real square matrix

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
0 & 2 & 3 \\
-1 & 1 & 2 \\
\end{bmatrix}.
\]

Laplacean developing along the 1\(^{st}\) column:

\[
\begin{vmatrix}
1 & -1 & 2 \\
0 & 2 & 3 \\
-1 & 1 & 2 \\
\end{vmatrix}
= 1 \cdot (-1)^{1+1} \begin{vmatrix}
2 & 3 \\
1 & 2 \\
\end{vmatrix} + 0 \cdot A_{21} - 1 \cdot (-1)^{2+1} \begin{vmatrix}
1 & 2 \\
-1 & 2 \\
\end{vmatrix}
= 4 - 3 - (-3 - 4)
= 1 + 7 = 8
\]

Laplacean developing along the 2\(^{nd}\) row:

\[
\begin{vmatrix}
1 & -1 & 2 \\
0 & 2 & 3 \\
-1 & 1 & 2 \\
\end{vmatrix}
= 0 \cdot A_{21} + 2 \cdot (-1)^{2+2} \begin{vmatrix}
1 & 2 \\
-1 & 2 \\
\end{vmatrix} + 3 \cdot (-1)^{2+3} \begin{vmatrix}
1 & -1 \\
-1 & 1 \\
\end{vmatrix}
= 2 \cdot (2 + 2) - 3 \cdot (1 - 1)
= 8
\]

Note. The Laplace's theorem states that the determinant of a square matrix is obtained by adding the products of the elements of a row [column] by their algebraic complements, and reduces the determinant calculation of a matrix of order "n" the calculation of determining the order of matrix “n-1”.

In applying the Laplace theorem, you should choose the row or column with the highest number of zeros (to facilitate the calculations).

Theorem. The Laplace Theorem has immediate consequence of the following properties:

1. Let A be a triangular matrix, then the determine of A is equal to the product of the diagonal elements;

2. Let A be a square matrix with a row [column] of zeros, then \(|A|=0\).

Theorem. Let \(A=[a_{ij}]\) be a square matrix of order n. Then:

1. If the matrix \(B\) is obtained from the matrix \(A\) by interchanging two rows[colums], then \(|A|=-|B|\), that is,

\[
A \rightarrow B \Rightarrow |A| = -|B|.
\]

(1) If the matrix \(B\) is obtained from the matrix \(A\), by multiplying a row[column] of \(A\) by a real scalar \(\lambda\neq 0\), then
\[ |B| = \lambda |A| \iff |A| = \frac{1}{\lambda} |B| , \]

that is,

\[
A \xrightarrow{L_i \rightarrow \lambda L_i \text{ or } C_j \rightarrow \lambda C_j} B \Rightarrow |A| = \frac{1}{\lambda} |B| .
\]

(2) if the matrix B is obtained from the matrix A, by replacing a given row [column] by their sum to another row [column] multiplied by a real scalar \( \lambda \), then \(|B| = |A| \), that is,

\[
A \xrightarrow{L_i \rightarrow L_i + \lambda L_k \text{ or } C_j \rightarrow C_j + \lambda C_l} B \Rightarrow |A| = |B| .
\]

Note. The combination of the last two theorems, can greatly improve the efficiency of calculation of determinant, compared with the method from the definition of computing determinant. That is, there may be used the elementary row operations [column] to obtain increasingly more zeros in a given row [column] of the matrix or transform matrix given a triangular matrix, before applying the Laplacean development and / or its consequences.

Exemple. Let the real square matrix be

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
0 & 2 & 3 \\
-1 & 1 & 2
\end{bmatrix} .
\]

Then:

\[
|A| = \begin{vmatrix}
1 & -1 & 2 \\
0 & 2 & 3 \\
-1 & 1 & 2
\end{vmatrix} = \begin{vmatrix}
1 & -1 & 2 \\
0 & 2 & 3 \\
0 & 0 & 4
\end{vmatrix} = 1 \cdot 2 \cdot 4 = 8 .
\]

Example. Calculate the determinant of the real square matrix

\[
A = \begin{bmatrix}
-1 & 1 & 1 & 2 \\
0 & 1 & -1 & 1 \\
-1 & 2 & 3 & 2 \\
1 & 2 & 3 & 1
\end{bmatrix} .
\]
Theorem (Properties of determinant). Let A be a square matrix of order $n \in \mathbb{N}$. Then:

1. $|\lambda A| = \lambda^n |A|$;
2. $|A^t| = |A|$;
3. If two rows [columns] of A are proportional, then $|A| = 0$;
4. If the row [column] of A can be broken down into the sum of two rows [columns], the value of the determinant of A is equal to the sum of the values of the determinants of two matrices in which this row [column] is used as portion and keeping the remaining rows [columns].

**Regular Matrix – Inverse of a Matrix**

Definition (Regular Matrix (or Invertible Matrix)). The real square matrix $A$, of order $n \in \mathbb{N}$, is said to be regular (or invertible), if there exist a real square matrix $B$, of the same order such that $AB = BA = I_n$. In this case, the matrix $B$ is said to be the inverse of $A$, and $B = A^{-1}$. Similarly, it is said that $A$ is the inverse of $B$, and it is written as $B = A^{-1}$.

Theorem. Let $A$ and $B$ be real square matrices of order $n$. then, $AB = I_n \Leftrightarrow BA = I_n$.

Note. By the previous theorem, the real square matrix $A$ of order $n$, is regular if there exist a real square matrix $B$, also of the order $n$, such that:

$AB = I_n$ or $A = I_n$.

Example. Let the real square matrix be

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$
Show that the matrix

\[
B = \begin{bmatrix}
0 & -1 & 1 \\
-1 & 0 & 1 \\
2 & 1 & -1
\end{bmatrix}
\]

is the inverse of A, that is, A^(-1)=B, equivalently B^(-1)=A.

Now, the previous note, we can just see that AB=I_3 or BA=I_3. Then,

\[
AB = \begin{bmatrix}
1 & -1 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -1 & 1 \\
-1 & 0 & 1 \\
2 & 1 & -1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Example. Let the real square matrix of order 2,

\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix},
\]

such that \(ad-bc\neq 0\), then,

\[
A^{-1} = \frac{1}{ad-bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}.
\]

Therefore,

\[
AA^{-1} = \frac{1}{ad-bc} \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix}
ad-bc & 0 \\
0 & ad-bc
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

**Theorem (Uniqueness of the inverse matrix).** Every regular matrix has one and only one inverse.

**Theorem (Properties of the inverse matrix).**

1. If A is a regular matrix, then A^(-1) is also a regular matrix and \(\| (A^\lambda) \|^\lambda=1\).

2. If A and B are regular matrices of the same order, AB is also a regular matrix and \(\| (AB)^\lambda \|^\lambda=1\).

3. If A is a regular matrix, then A^t is also a regular matrix and \(\| (A^t)^\lambda \|^\lambda=1\).

4. If A is a regular matrix, then for a real scalar \(\lambda\neq 0\), \(\lambda A\) is also a regular matrix and \(\| (\lambda A)^\lambda \|^\lambda=1\).

5. Let \(k\in N\) and let \(A_i\), \(i=1,2,3,...,k\), be a regular matrices then \(A_1 A_2 A_3...A_{(k-1)} A_k\) is also a regular matrix and \(\| (A_1 A_2 A_3...A_k)\|^\lambda=1\).

6. Let \(k\in N\). If A is a regular matrix then A^k is also a regular, and \(\| (A^k)\|^\lambda=1\).
Method for determining the inverse of a regular matrix

In the following case, I will present one of the methods that can be used to calculate the inverse of a regular matrix.

Let the regular real square matrix of order 3 be:

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

Its inverse \(B=A^{-1}\), is also a square matrix of order 3, then:

\[
A^{-1} = B = \begin{bmatrix}
a & b & c \\
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2
\end{bmatrix}.
\]

As the matrix of coefficients gives rise to system (1), (2), and (3) (in this case they are equal to \(A\) then) the three systems can be solved simultaneously:

\[
AA^{-1} = I_3
\]

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
a & b & c \\
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
a & a_1 + a_2 \\
a & a_1 + a_2 \\
a & a_1 + a_2
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix} \\
\begin{bmatrix}
b + b_1 + b_2 \\
b - b_1 + b_2 \\
b + b_1
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \\
\begin{bmatrix}
c + c_1 + c_2 \\
c - c_1 + c_2 \\
c + c_1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

\[
\begin{cases}
a + a_1 + a_2 = 1 \\
a - a_1 + a_2 = 0 & (1) \\
a + a_1 = 0 & (2)
\end{cases}
\]

Therefore,

\[
B = A^{-1} = \begin{bmatrix}
-1/2 & 1/2 & 1 \\
1/2 & -1/2 & 0 \\
1 & 0 & -1
\end{bmatrix}.
\]

Method of determining the inverse of a regular matrix \(A\) of order \(n\). The method consists of reducing the matrix \([\text{All}_n\ |\ A]\) reduce row echelon form, \([\text{I}_n \ |\ A^{-1}]\), using elementary row operation.

\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 1 & 0 \\
\end{bmatrix}
\xrightarrow{L_2 \rightarrow L_2 - 2L_1}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -2 \\
\end{bmatrix}
\xrightarrow{L_2 \rightarrow L_2 + L_1}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
\xrightarrow{L_2 \rightarrow L_2 - L_1}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 2 \\
\end{bmatrix}
\]

Example. Determine the inverse of the regular real matrix \(A=[1\ 1\ 2\ 1]\).

Then, \(A^{-1}=[-1\ 1\ 2\ -1]\).
Theorem (Criterion of regular matrix). Let \( A \) be a real square matrix real of order \( n \). Then, the following statement is true:

1. \( A \) is regular;
2. \( c(A)=n; \)
3. \( |A|\neq0; \)
4. The reduce row echelon form obtain from \( A \), by successive application of elementary row operation is the identity matrix \( I_n \).
5. The system of linear equation \( AX=B \) can be determine for any column matrix \( B \), \( AX=B\Rightarrow X=A^{-1} B \).

Theorem (Determinant of a regular matrix). Let \( A \) be a square regular matrix of order \( n \). Then, \( |A^{-1}|=1/(|A|) \).

Definition (Adjoint). Let \( A \) be a real square matrix order \( n \). The complementary of the matrix \( A \) is the matrix \( C=[A_{ij}], \text{ } i,j\in\{1,n\} \), where \( A_{ij} \) are the entries of the cofactors \( a_{ij} \) of the matrix \( A \). The matrix \( \text{Adj}(A)=C^t \) is called the Adjoint of \( A \).

Theorem. Let \( A \) a square matrix order \( n \). Then:

1. \( A\cdot\text{Adj}(A)=\text{Adj}(A)\cdot A=|A|I_n \);
2. If \( A \) is a real regular matrix, then \( |A|\neq0 \), So

\[
A \cdot \left( \frac{1}{|A|} \cdot \text{Adj}(A) \right) = I_n \iff A^{-1} = \frac{1}{|A|} \cdot \text{Adj}(A).
\]

Example. Let the real square matrix of order 3,

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
0 & 2 & 3 \\
-1 & 1 & 4 \\
\end{bmatrix}.
\]

Find the adjoint matrix of the matrix \( A \):

\[
A_{11} = \begin{vmatrix}
1 & 3 \\
-1 & 4 \\
\end{vmatrix} = 5 \quad A_{12} = \begin{vmatrix}
0 & 3 \\
-1 & 4 \\
\end{vmatrix} = -3 \quad A_{13} = \begin{vmatrix}
0 & 2 \\
-1 & 4 \\
\end{vmatrix} = 2
\]

\[
A_{21} = \begin{vmatrix}
2 & 3 \\
-1 & 4 \\
\end{vmatrix} = 6 \quad A_{22} = \begin{vmatrix}
1 & 3 \\
-1 & 4 \\
\end{vmatrix} = 6 \quad A_{23} = \begin{vmatrix}
1 & 2 \\
-1 & 4 \\
\end{vmatrix} = 0
\]

\[
A_{31} = \begin{vmatrix}
2 & 3 \\
1 & 4 \\
\end{vmatrix} = -7 \quad A_{32} = \begin{vmatrix}
0 & 3 \\
1 & 4 \\
\end{vmatrix} = -3 \quad A_{33} = \begin{vmatrix}
0 & 2 \\
1 & 4 \\
\end{vmatrix} = 2
\]

\[
\text{Adj} \ (A) = C^t = \begin{bmatrix}
5 & -3 & 2 \\
6 & 6 & 0 \\
-7 & -3 & 2 \\
\end{bmatrix}^t = \begin{bmatrix}
5 & 6 & -7 \\
-3 & 6 & -3 \\
2 & 0 & 2 \\
\end{bmatrix}
\]
However, $|A| = 8 + 3 - (-4 + 3) = 12$

Note that:

$A \cdot \text{Adj} (A) = \text{Adj} (A) \cdot A = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} = 12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3$.

Since $A$ is regular, we have:

$A^{-1} = \frac{1}{|A|} \text{Adj} (A) = \frac{1}{12} \begin{bmatrix} 5 & 6 & -7 \\ -3 & 6 & -3 \\ 2 & 0 & 2 \end{bmatrix}$.

Theorem (Determinant of a product). Assuming that operation between square matrices shown are possible, then:
1. $|A \cdot B| = |A| \cdot |B|$;
2. $|A_1 \cdot A_2 \cdots A_k| = |A_1| \cdot |A_2| \cdots |A_k|$, for all $k \in \mathbb{N}$;
3. $|A^k| = (|A|)^k$, for all $k \in \mathbb{N}$.

Definition (Cramer’s Rule). A system of linear equations $AX = B$, with $n$ unknowns, is said to be Cramer’s rule if the matrix $A$ is invertible.

Example. It is known that the real matrix

$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ -1 & 1 & 4 \end{bmatrix}$

is regular, and its inverse is

$A^{-1} = \frac{1}{12} \begin{bmatrix} 5 & 6 & -7 \\ -3 & 6 & -3 \\ 2 & 0 & 2 \end{bmatrix}$.

Thus, the system of linear equations

$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

is the Cramer’s Rule. Then,
Theorem. Let $AX=B$ be a Cramer’s rule with $n$ unknowns. It is represented by the matrix $A_i(B)$ obtain from matrix $A$ by replacing the column $i$ the column of independent terms $B$. Then:

$$x_i = \frac{|A_i(B)|}{|A|},$$

for all $i \in \{1, 2, 3, \ldots, n\}$.

Exemple. Determine $x_2$ by using the Cramer’s rule

$$AX = B \Leftrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

$$x_2 = \frac{|A_2(B)|}{|A|} = \frac{1}{12} \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ -1 & 3 & 4 \end{vmatrix} = \frac{6}{12} = \frac{1}{2}.$$

Conclusion

Determinant of any square matrix can be calculated. In calculating the determinant of a square matrix, one can use different method. However, elementary row operations (on rows and/or columns) have great important in determinant calculation by using the properties.

Determinant of a square matrix allows among other things identifying regular matrix and determine its inverse, and solve using cramer’s rule.

In the use of Cramer’s rule, $AX=B$, the matrix $A$ has to be regular, we have $AX=B \Leftrightarrow X = A^{-1} B$.

Assessment

1) Determine the value of the expression shown below, considering that $A$, $B$ and $C$ are real square matrices of order $n$, and $|A|=-2$, $|B|=3$ and $|C|=-1$:

$$\left| B^t \cdot A^{-1} \cdot B^{-1} \cdot C \cdot A^2 \cdot (C^{-1})^t \right|$$

(2) Considering that the matrix $A$ is a square matrix of order 3, and
\[ |2A^{-1}| = 5 = |A^2 \cdot (B')^{-1}|, \]

compute \(|A|\) and \(|B|\).

\[ A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 1 & 2 & -2 & 3 \\ -1 & 2 & 3 & 3 \\ 2 & 3 & 4 & 0 \end{bmatrix}. \]

(3) Find the determinant of the following real square matrix of order 4:

\[ \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -1, \]

Calculate \(A^\wedge\wedge(-1)\)

(4) Consider

\[ \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -1, \]

Compute

\[ \begin{vmatrix} -2a & -2b & -2c \\ 2p + x & 2q + y & 2r + z \\ 3x & 3y & 3z \end{vmatrix}. \]

(5) Solve the following equation:

\[ \begin{vmatrix} 0 & x - 2 & 0 & 0 \\ x - 1 & 0 & x & 0 \\ 0 & x & 0 & x - 2 \\ 0 & 0 & x - 1 & 0 \end{vmatrix} = 0. \]

(6) Consider the real square matrix

\[ A = \begin{bmatrix} 3 & 4 & -1 \\ 0 & 5 & -4 \\ 8 & -6 & 2 \end{bmatrix}. \]

Determine \(|A|\) and \(\text{Adj}(A)\);

Show that \(A\cdot\text{Adj}(A)=|A|I_3\);

Determine \(A^\wedge\wedge(-1)\).
(7) Consider the following linear system:

\[
\begin{cases}
   x_1 + x_3 = 1 \\
   x_2 + x_4 = -1 \\
   x_1 + 2x_3 + x_4 = -1 \\
   x_1 - x_2 + x_3 = 1
\end{cases}
\]

**Represent the system in the form\( AX=B \);**

**Determine** \( x_1, x_2, x_3 \) and \( x_4 \) **by using cramer’s rule.**

Show that \( X=A^{-1} B \).

**1.4 – SCILAB Software: application in Linear Algebra.**

**Introduction**

SCILAB is a software for scientific and visualization with open source and is free. For more information about this powerful tool for scientific computation, you can visit the official website: www.scilab.org.

In this activity, we will use some important SCILAB tool useful for computing in the context of linear algebra, such as :

- Matrix Operations (including determinant and inverse);
- Solving system of linear equation;
- Operations with vectors in a vector space \( n \in \mathbb{N} \) and \( n>1 \);
- Calculating eigenvalues and eigenvectors of a matrix (or endomorphism).

The idea is to show the basic tools for performing some calculations, mainly connected with matrices since the rest is summarise in real matrix.

For more information, the reader is referred to the Linear Algebra tools for help:
It is used to facilitate calculations which are very difficult.

In this activity, you use the following SCILAB: “scilab-5.4.0” (64 bit).

**Activity Details**

Matrices and their operations

Only to make the notes simple, I will represent a matrix \(A=\begin{bmatrix} a_{ij}\end{bmatrix}_{m \times n}\), \(m, n \in \mathbb{N}\), by \(A_{m \times n}\), in SCILAB by \(A_{mn}\).

Writing of matrix: The writing of matrix in SCILAB used “,” (comma) or “ ” (blank space) to mark the end of an entry and “;” to mark the end of a row. Entries are placed in a row.

Example. Write the real square matrix of dimension 4,

\[
A = \begin{bmatrix}
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 \\
\end{bmatrix}
\]
Example. Write the real matrix $A = [a_{ij}]$, $i \in \{1,2,3,4\}$, $j \in \{1,2,3\}$ e $a_{ij} = 1/(i+j-1)$.

Example. Write the matrix $B = [b_{ij}]$, $i,j \in \{1,2,3\}$ e $b_{ij} = i-j+1$.

Example. Let the real matrices be $A = [1 \ 2 \ -1 \ 0 \ -3 \ 4]$ and $B = [-1 \ 5 \ 7 \ -1 \ 3 \ -8]$.

(a) Determine $-3A + 2B - A$
Example. Determine the product $AB$, where $A=\begin{bmatrix} -1 & 2 & -2 & 0 & 1 & 3 \\ -2 & 3 & 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 & 1 & 3 \\ -2 & 3 & 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 1 & 3 & 0 \\ 1 & 2 & 3 & 0 & 1 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 2 & 0 & 3 & 1 & 2 & -2 & 1 & -2 & 3 \end{bmatrix}$.

Example. Find the determinant of the real square matrix

$$A = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ -1 & 2 & 3 & 2 \\ 1 & 2 & 3 & 1 \end{bmatrix}.$$ 

In $[e,m]$, $m$ is the mantissa of the determinant of $A$, of base 10; and $e$ is an integer representing the exponent 10, when the determinant of $A$ is represented in base 10. That is, $|A|=-7$.

Example. Calculate the inverse of the real square matrix

$$A = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ -1 & 2 & 3 & 2 \\ 1 & 2 & 3 & 1 \end{bmatrix}.$$
Creating identity matrix \(I_{nn} = \text{eye}(n,n)\), \(n \in \mathbb{N}\). If a square matrix \(A\) had been inserted into the “console”, then \(\text{eye}(A)\).

Symmetric and Skew symmetric Matrix.

Transpost of \(A\), \(A^t\), in SCILAB is \(A'\).

Example. Let the real square matrix be \(A = \begin{bmatrix} 1 & -2 & 2 & 2 & 1 & -2 & -1 & 1 & 3 \end{bmatrix}\). Determine a symmetric matrix \(B\), and a skew symmetric matrix \(C\), such that \(B + C = A\).

\[\begin{align*}
\text{Transpost of } A, \quad A^t, \text{ in SCILAB is } A'.
\end{align*}\]

Cramer’s Rule

Example. Solve for \(x_2\) by using the Cramer’s rule

\[
AX = B \Leftrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.
\]
Conclusion

With SCILAB, we can compute various operations under Linear Algebra, matrix operations, calculation of eigenvalues and eigenvectors of a matrix (or endomorphism), etc.

Its use does not relieve the mental calculation, but help facilitate some calculations.

Summary

In relation to operation on matrices, one can add matrices with equal dimensions, but you can only multiply matrices in which the length of the column is equal to the length of row.

The general method for solving a system of linear equation is the scale of the augmented matrix. However, the system of linear equation, matrices with constant coefficient can be solve using Cramer’s rule.

One can determine the inverse of a matrix by solving the system and by adjoint matrix (application of determinant of square matrix).

Note that invertible matrix are necessarily square. And also you can only calculate the determinant of square matrix.

SCILAB has more than enough tools to solve problems in the field of linear algebra. It can be used to solve difficult problems and solve problem that can’t be solve mentally.


**Unit Assessment**

Check your understanding!

Assessment of the unit system of linear equation. Matrix and determinant

Instructions

The evaluation test has seven question with some paragraph.

Answer each question clearly and justify each step of solution.

**Grading Scheme**

Each question carried 10 marks. I will approved passes for the student who has 50% of the marks.

**Feedback**

(1) Discuss the rank of following the following matrix base on the parameter of $\alpha$ and $\beta$:

$$A = \begin{bmatrix} 2 & 1 & \alpha & 1 \\ 1 & 1 & \alpha & \alpha/2 \\ 1 & \beta & \alpha & 1 \end{bmatrix}.$$  

(2) Show that the real square matrix

$$A = \begin{bmatrix} 1 & 7 \\ 0 & 4 \end{bmatrix}$$

is the root of the polynomial $f(x)=x^2-5x+4$.

(3) Let $A$ and $B$ be a real regular matrices. Prove that if $A+B$ is regular, then

$$(A^{-1} + B^{-1})^{-1} = A \cdot (A + B)^{-1} \cdot B.$$  

(4) Calculate the inverse of the real square matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{bmatrix}.$$  

(5) Determine the real square matrix $A$, given that

$$(A^{-1} - 3I)^t = 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$  


(6) Discuss the following system of linear equation base on the value of a and b;

\[
\begin{align*}
ax + y - z + aw &= 0 \\
(a + 1)y + z + w &= 1 \\
-x + y + (a + 1)w &= b
\end{align*}
\]

(a) Solve for a=-1 and b=-3.

(7) Calculate the determinant of the following real square matrix:

\[
A = \begin{bmatrix}
1 & -2 & 0 & 3 \\
1 & 2 & -2 & 3 \\
-1 & 2 & 3 & 3 \\
2 & 3 & 4 & 0
\end{bmatrix}.
\]

(8) If

\[
\begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{vmatrix} = 7,
\]

verify the value of the following determinant:

\[
C = \begin{bmatrix}
a_1 - 5c_1 & a_2 - 5c_2 & a_3 - 5c_3 \\
10b_1 & 10b_2 & 10b_3 \\
-4c_1 & -4c_2 & -4c_3
\end{bmatrix}.
\]

(9) Consider the real square matrices

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 5 \\
3 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & 2 & 2 \\
0 & 2 & 2 \\
1 & 1 & 2
\end{bmatrix}.
\]

**Unit Readings and Other Resources**

The readings in this unit are to be found at course level readings and other resources.

- LAY, D.C., Linear Algebra and its Application, Addison-Wesley Production Company, 1994
- [https://en.wikipedia.org/wiki/Linear_algebra](https://en.wikipedia.org/wiki/Linear_algebra)
Unit 2. Real Vector Space

Unit Introduction

In this chapter we first develop the notion of 2-vectors and 3-vectors along with their properties very carefully and systematically. A good mastery of this basic tool will be helpful in understanding n-vectors.

Unit Objectives

Upon completion of this unit you should be able to:

• discuss a vector space and a subspace of a given vector space
• identify linearly dependent and linearly independent vectors.
• construct equivalent vector systems.
• determine a subspace spanned by a set of vectors.
• represent a vector subspace through a homogeneous system and through their generators.
• determine the sum of the intersection and the meeting of vector subspaces.
• identify a direct sum.
• determine a complementary subspace of a particular vector subspace.
• determine the basis and dimension of vector space finitely generated and of its vector subspace.
• identify an inner product.
• determine orthonormal basis for a given vector space.
• operate the cross product of two vectors.
Key Terms

Vector Space: Let be a nonempty set and be a field. It is said that is a vector space on the field when:

is defined an operation called addition written as “+”, such that,

∀ u , v ∈ V, u + v ∈ V,

and goes with the following properties:

∀ u , v ∈ V, u + v = v + u ;
∀ u , v , w ∈ V, u + (w + v ) = (u + w ) + v ;
there exist a vector in V called “zero vector”, written as 0 , such that
∀ u ∈ V, u + 0 = u ;
∀ u ∈ V, there exist a vector in V, that is called “symmetric of u”, and that is written as “-u”, such that u + (-u ) = 0 .

A product (or multiplication), “denoted by “product (or multiplication) by scalar”, such that

∀ u , v ∈ V, ∀ λ ∈ F, λ u ∈ V,

and goes with the following properties:

∀ λ ∈ F, ∀ u , v ∈ V, we have λ( u + v ) = λ u + λ v ;
∀ λ, β ∈ F, ∀ u ∈ V, we have (λ + β )u = λ u + β u ;
∀ λ, β ∈ F, ∀ u ∈ V, we have (λ β ) u = λ (β u ) ;
∀ u ∈ V, we have 1 u = u .

When these condition are satisfied, the elements of V denominated by vectors, and that of F by scalars.

If the field F=R the vector space is said to be real.

Linear Dependence and Linear Independence: Let V be a real vector space. It is said that:

The vectors v , v , ..., v ∈ V are linearly independent if λ v , , + λ v , + ... + λ v = 0 ,
then λ = λ = ... = λ = 0 , that is, the unique zero linear combination of v , v , ..., v is the trivial zero linear combination.
The vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V \) are linearly dependent if there exist a real scalar \( \lambda_1, \lambda_2, \ldots, \lambda_k \), not all zero, such that
\[
\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k = \mathbf{0},
\]
that is, beyond trivial zero linear combination, there exist another linear combination \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \).

By convention, the zero vector space \( V = \{ \mathbf{0} \} \) is a basis of the empty set \( \emptyset \).

Euclidean Space: A real vector space \( V \), with inner product and finite dimension is denoted by euclidean space.

Orthonormal Basis: Let \( V \) be a euclidean space of dimension \( n \in \mathbb{N} \). It is said that:

A basis \( B \) of \( V \) is orthogonal if \( B \) is an orthogonal system of vectors.

A basis \( B \) of \( V \) is orthonormal if \( B \) is an orthonormal system of vectors.

Subspace: Let \( V \) be a real vector subspace and \( S \subseteq V \). It is said that \( S \) is a vector subspace of \( V \), and is denoted by \( S \subseteq V \) if:

\[ S \neq \emptyset \text{ (empty set)}; \]
\[ \forall \mathbf{u}, \mathbf{v} \in S, \text{ we have } \mathbf{u} + \mathbf{v} \in S; \]
\[ \forall \lambda \in \mathbb{R}, \forall \mathbf{u} \in S, \text{ we have } \lambda \mathbf{u} \in S. \]

Learning Activities

Activity 2.1 - Subspace vector: definition and properties; operations

Introduction

This activity is defined vector subspace and gives up the main emphasis on subspaces of vector spaces finitely generated. It is up a subspace in various forms and make some operations with subspaces, namely the sum, the intersection and the union.

It introduces the concept of direct sum, which is very important in determining a complementary subspace of a certain vector subspace finitely generated.

Fill the dimensions of the theorem that relates the dimensions finitely generated two subspaces with dimension of their sum.
Definition (vector subspace). Let V be a real vector space and \( S \subseteq V \). It is said that \( S \) is a vector subspace \( V \), and is represented by \( S \leq V \) that:

1. \( S \neq \emptyset \) (empty set);
2. \( \forall \mathbf{u}, \mathbf{v} \in S \), \( \mathbf{u} + \mathbf{v} \in S \);
3. \( \forall \lambda \in \mathbb{R}, \forall \mathbf{u} \in S \), \( \lambda \mathbf{u} \in S \).

Theorem (Criterion vector subspace). Let V be a real vector space and \( S \subseteq V \). It is said that \( S \) is a vector subspace of \( V \) if:

1. \( S \neq \emptyset \);
2. \( \forall \lambda, \beta \in \mathbb{R}, \forall \mathbf{u}, \mathbf{v} \in S \), \( \lambda \mathbf{u} + \beta \mathbf{v} \in S \).

Theorem. Let V be a real vector space and \( S \leq V \). So:

1. \( \mathbf{0} \in S \);
2. \( \mathbf{u} \in S \), then \( -\mathbf{u} \in S \);
3. \( S \) is a real vector space, in relation to the restrictions of the operations defined on \( V \) the set \( S \).

Example. Let V be a real vector space arbitrary. So \( V \leq V \) (improper subspace \( V \)) and \( \{0 \} \leq V \) (null subspace of \( V \)).

Example. In the real vector space \( \mathbb{R}^3 \), considers the subset

\[ S = \{(a,b,0):a,b \in \mathbb{R}\} \] (Oxy plane, the equation of plane \( z=0 \)).

Note that \( S \leq \mathbb{R}^3 \) because:

1. \( S \neq \emptyset \), since the zero vector \((0,0,0)\in S \) (just take \( a=b=0 \));
2. Let \( \mathbf{u} = (a_1,b_1,0) \), \( \mathbf{v} = (a_2,b_2,0) \) arbitrary vector \( S \) (i.e., \( a_1, a_2, b_1 \) e \( b_2 \) are arbitrary real) and arbitrary scalar \( \lambda \):
3. \( \mathbf{u} + \mathbf{v} = (a_1+a_2,b_1+b_2,0) \in S \), because the sum of two real numbers is a real number;
   \( \lambda \mathbf{u} = (\lambda a_1,\lambda a_2,0) \in S \), as the product of two real numbers is a real number.

Example. In \( P^2 \) vector space, we consider the subset

\[ S = \{ax^2+bx+1:a,b \in \mathbb{R}\} \]

Note that \( S \leq P^2 \). Just see that the zero vector, the zero \( (0) \) does not belong to \( S \).

Otherwise, one can easily see that the sum of any two vectors \( S \) does not belong to \( S \), or the dot product of any real number \( \lambda \neq 1 \) by a vector \( S \) does not belong to \( S \).

Remark. Given a real vector space \( V \) and its subset \( S \). If there are two vectors \( S \) the sum of which is not \( S \). Or if the dot product of a given real number for a given vector \( S \) not in \( S \). Or if the zero vector is not \( S \), then \( S \leq V \).
Theorem. Let \( V \) be a real vector space finitely generated, and is \( S \subseteq V \). So:

1. \( S \) is finitely generated;
2. \( \dim(S) \leq \dim(V) \);
3. \( \dim(S) = \dim(V) \) if and only if, \( S = V \).

Example. The vector subspace of \( \mathbb{R}^3 \), \( S = \{ (a,b,0) : a, b \in \mathbb{R} \} \), has a dimension of 2. Indeed,

\[
S = \{ (a,0,0) + (0,b,0) : a, b \in \mathbb{R} \} = \{ a(1,0,0) + b(0,1,0) : a, b \in \mathbb{R} \} = \langle (1,0,0),(0,1,0) \rangle
\]

As the vectors \( (1,0,0) \) and \( (0,1,0) \) generate the vector subspace \( S \) and are linearly independent, then \( S \) constitute a base, then \( \dim(S) = 2 \).

Definition (subspace generated). Let \( V \) be a real vector space, and let \( X = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \subseteq V \). The set \( S \) consists of all linear combinations of the vectors \( X \), or

\[
S = \{ \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \cdots + \lambda_k \vec{v}_k : \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R} \}
\]

is a vector subspace \( V \)-called subspace spanned by \( X \), is represented by \( S = \langle X \rangle \). By convention, \( \langle \emptyset \rangle = \{0\} \).

Example. In the real vector space \( P^2 \), is \( X = \{ x^2 - x + 1, -2x^2 - 3x \} \). The vector subspace generated by \( X \) is:

\[
S = \langle X \rangle = \{ \alpha(x^2 - x + 1) + \beta(-2x^2 - 3x) : \alpha, \beta \in \mathbb{R} \} = \{ (\alpha - 2\beta)x^2 + (-\alpha - 3\beta)x + \alpha : \alpha, \beta \in \mathbb{R} \}
\]

Theorem. The solution set for a system of linear homogeneous equations with real coefficients and \( n \) unknowns is a vector subspace of the real vector space \( \mathbb{R}^n \).

Example. Consider the following system of linear equations for the body \( R \):

\[
\begin{align*}
x - 2y + z - w &= 0 \\
x + y + 2z - w &= 0
\end{align*}
\]

As the system of homogeneous linear equations has four unknowns, then the whole solution is a vector subspace of \( \mathbb{R}^4 \).
Determine this vector subspace:

\[ \mathbf{S} = \{(5\alpha - 3\beta, 3\alpha - 2\beta, \alpha, \beta) : \alpha, \beta \in \mathbb{R}\} = \langle (5, 3, 1, 0), (-3, -2, 0, 1) \rangle \]

is a subspace \( \mathbb{R}^4 \) vector of dimension 2.

Theorem. Let \( V \) be a vector space of dimension \( n \), and let \( \mathbf{B} = \{ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \} \) be its base. Let \( \mathbf{S} \) be a vector subspace of \( V \) with dimension \( k \). Then, it is possible to constitute a system of \( "n-k" \) linear homogeneous equations with real coefficients, whose solutions are components of the vectors \( \mathbf{S} \) in relation to the base \( \mathbf{B} \) to \( V \), initially considered.

Remark. The above theorem states that any subspace of a finite dimensional vector space can be represented by a system of linear homogeneous equations with respect to a certain basis.

Example. In the real vector space \( \mathbb{P}^2 \) it is considered the vector subspace \( \mathbf{S} \) defined by the homogeneous system

\[ x - y + z = 0, \]

in relation:

In the canonical basis.

In the base \( \mathbf{B} = \{ x^2 - x, x - 1, x^2 + 2 \} \).

Determine \( \mathbf{S} \) and a basis.

In this case, first, solves the given system.

\[ x - y + z = 0 \iff \begin{cases} x = \alpha - \beta \\ y = \alpha \in \mathbb{R} \\ z = \beta \in \mathbb{R} \end{cases} \]

For the first paragraph (a)),

\[ \mathbf{S} = \{ (\alpha - \beta)x^2 + \alpha x + \beta : \alpha, \beta \in \mathbb{R} \} = \langle x^2 + x, -x^2 + 1 \rangle \]

As vectors \( x^2 + x \) e \( -x^2 + 1 \) generate \( \mathbf{S} \) and are linearly independent, then \( \mathbf{B} = \{ x^2 + x, -x^2 + 1 \} \) is a basis \( \mathbf{S} \).

For the second subparagraph (b)),

\[ \mathbf{S} = \{ (\alpha - \beta)(x^2 - x) + \alpha(x - 1) + \beta(x^2 + 2) : \alpha, \beta \in \mathbb{R} \} = \langle x^2 - 1, x + 2 \rangle \]

As the vectors \( x^2 - 1 \) e \( x + 2 \) are linearly independent and generate \( \mathbf{S} \), then \( \mathbf{B} = \{ x^2 - 1, x + 2 \} \) is a basis \( \mathbf{S} \).

Remark. Alternatively, one may define the vector subspace \( \mathbf{S} \) of the previous example, as follows:

\[ \mathbf{S} = \langle a(x^2 + bx + c): a - b + c = 0 \rangle. \]

\[ \mathbf{S} = \langle a(x^2 - x) + b(x - 1) + c(x^2 + 2): a - b + c = 0 \rangle. \]
To determine if $S$ proceeds in the same fashion, i.e., it solves the homogeneous system data, and it replaces the corresponding variables in the generic vector $S$, namely:

$$a - b + c = 0 \iff \{a = \alpha - \beta, b = \alpha \in \mathbb{R}, c = \beta \in \mathbb{R}\}$$

Soon,

$$S = \{(\alpha - \beta)x^2 + \alpha x + \beta : \alpha, \beta \in \mathbb{R}\} = (x^2 + x, -x^2 + 1).$$

$$S = \{(\alpha - \beta)(x^2 - x) + \alpha(x - 1) + \beta(x^2 + 2) : \alpha, \beta \in \mathbb{R}\} = (x^2 - 1, x + 2).$$

Example. In the real vector space $P^2$ is $S = (x^2 - 1, x + 2)$. Representing $S$ through a homogeneous system, in relation to:

- Standard basis;
- Base $B = \{x^2 - 1, x - 1, -x^2 + x + 1\}$

For the first paragraph (a), it is necessary to determine the condition in the variables $a, b$ such that

$$\alpha(x^2 - 1) + \beta(x + 2) = ax^2 + bx + c,$$

that is, the condition for which $S$ vectors are linear combinations of the standard basis vector $P^2$

\[
[A\mid B] = \begin{bmatrix}
1 & 0 & a \\
0 & 1 & b \\
-1 & 2 & c
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & a \\
0 & 1 & b \\
0 & 0 & \alpha - 2b + c
\end{bmatrix}
\]

So,

$$S = \{ax^2 + bx + c : a - 2b + c = 0\}.$$

For the second subparagraph (b), it is necessary to determine the condition in the variables $a, b$, and $c$ such that

$$\alpha(x^2 - 1) + \beta(x + 2) = a(x^2 - 1) + b(x - 1) + c(-x^2 + x + 1),$$

that is, the condition for which $S$ vectors are linear combinations of the vectors of the base $B$ of $P^2$.

\[
\alpha(x^2 - 1) + \beta(x + 2) = a(x^2 - 1) + b(x - 1) + c(-x^2 + x + 1)
\]

$$\iff \alpha(x^2 - 1) + \beta(x + 2) = (a-c)x^2 + (b+c)x - a + b + c$$

\[
[A\mid B] = \begin{bmatrix}
1 & 0 & a - c \\
0 & 1 & b + c \\
-1 & 2 & -a - b + c
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & a - c \\
0 & 1 & b + c \\
0 & 0 & -3b - 2c
\end{bmatrix}
\]
So, \( S = \{a(x^2-1)+b(x-1)+c(-x^2+x+1) : -3b-2c=0\} \)

Example. In the real vector space \( \mathbb{R}^3 \), be the vector subspace \( S = \langle (1,-1,2) \rangle \). Representing \( S \) through a homogeneous system, in relation to:

**Canonical basis.**

\[ B = \{(1,1,1),(0,1,-1),(-1,-1,0)\}. \]

To point a), it is necessary to determine the condition in the variable \( x, y \) and \( z \) such that \( \alpha(1,-1,2) = (x,y,z) \)

So, \( S = \{(x,y,z) : x+y=-2x+z=0\} \)

For b), it is necessary to determine the condition in the variable \( x, y \) and \( z \) such that

\[ \alpha(1,-1,2) = x(1,1,1)+y(0,1,-1)+z(-1,-1,0) \]

\[ \Leftrightarrow \alpha(1,-1,2) = (x-z, x+y-z, x-y) \]

So, \( S = \{(x-z, x+y-z, x-y) : 2x+y-2z=-x-y+2z=0\} \)

Theorem (subsets of properties generated). \( V \) is an arbitrary real vector space, and \( X \) and \( Y \) are subsets of \( V \). Then any:

1. \( X \subseteq \langle X \rangle \).
2. \( X = \langle X \rangle \) if and only if, \( X \subseteq V \).
3. \( \langle V \rangle = V \).
4. \( \langle \{0^+\} \rangle = \{0^-\} = \{0^+\} \).
5. If \( X \subseteq Y \), then \( \langle X \rangle \subseteq \langle Y \rangle \).
6. If \( S \subseteq V \) and \( X \subseteq S \), then \( \langle X \rangle \subseteq S \).
7. \( \langle X \rangle \) is the only vector subspace of \( V \) that satisfies the properties 1 and 6.
Exercises.

1. Check which of the following subsets are vector subspaces of vector spaces indicated, indicating for each of these a base:
   a. \( S=\{(x,y,z,w) : x=0 = y+z\} \) in \( \mathbb{R}^4 \)
   b. \( S=\{ax^3+bx^2+cx+d:a \text{ is irrational}\} \) in \( \mathbb{P}^3 \)
   c. \( S=\{(x,y,z,w) : x+y=2z-w=0\} \) in \( \mathbb{R}^4 \)
   d. \( S=\{p(x) : p(0)=0\} \) in \( \mathbb{P}^5 \)
   e. \( S=\{ax^2+bx+c : b\geq 0\} \) in \( \mathbb{P}^2 \)
   f. \( S=\{(x,y,z,w) : xy=0\} \) in \( \mathbb{R}^4 \)
   g. \( S=\{(x,y,z,w) : |x|\geq 2\} \) in \( \mathbb{R}^4 \)
   h. \( S=\{ax^2+(a-b)x+b : a,b \in \mathbb{R}\} \) in \( \mathbb{P}^2 \)

2. Find a basis for each of the vector subspaces of vector spaces indicated:
   a. \( S=\{(x,y,z,w) : x+2y-z=x+y+2w=y-z+w=0\} \) in \( \mathbb{R}^4 \)
   b. \( S=\{a(x^2-x)+b(x^2+1)+c(x+2) : a-b+2c=0\} \) in \( \mathbb{P}^2 \)

3. Set by means of linear homogeneous equations, each vector subspaces of the real vector space \( \mathbb{P}^3 \), relative to the base indicated:
   a. \( S=\langle x^2-1 \rangle \), relative to the base \( B=\{x^2-x,x^2+1,x+2\} \).
   b. \( S=\langle x^2-1 \rangle \) in relation to the standard basis.
   c. \( S=\langle x^2-1,x+1 \rangle \) relative to the base \( B=\{x^2-x+1,x-1,-x+2\} \).
   d. \( S=\langle x^2-x,x+1 \rangle \) with respect to the standard basis.

4. Determine the vector subspace of \( \mathbb{P}^3 \) defined by the system \( \{x-y+z+2w=0 \ x-2y+z+w=0\} \) relative to the base \( B=\{x^3-x^2,x^2-x,x-1,x^3\} \)

5. Determine the vector subspace of \( \mathbb{R}^4 \), defined by the system \( \{x-y+z+2w=0 \ x-2y+z+w=0\} \)

6. Determine a base and the dimension of each vector subspaces of vector spaces indicated:
   a. \( S=\{a(x^3-x^2)+b(x^2-x)+c(x-1)+dx^3 : a-b=a+b-c+2d=0\} \) in \( \mathbb{P}^3 \)
   b. \( S=\{x(1,1,1)+y(0,-2,1)+z(-1,-3,1) : x-y=y-z=0\} \) in \( \mathbb{R}^3 \)
   c. \( S=\{(3x+y,y+z,z,0) : x,y,z \in \mathbb{R}\} \) in \( \mathbb{R}^4 \)
   d. \( S=\{ax^3+(-a+b)x^2+(-b+c+2d)x-c-2d : a,b,c,d \in \mathbb{R}\} \) in \( \mathbb{P}^3 \)

Operations with vector subspaces

**Theorem (Intersection of subspaces)**. Let \( V \) be a real vector space, and are \( F,G \leq V \). so \( F \cap G \leq V \).

**Remark**. Let \( V \) be a real vector space \( n \in \mathbb{N} \) size, and are \( F,G \leq V \). Given a base \( B=\{e_1,e_2,...,e_n\} \) of \( V \), \( F \) and \( G \) are represented with respect to this base by
respectively, then the vector subspace \( F \cap G \) is represented on the same basis by

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\
  \vdots \\
  a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n &= 0 \\
  b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n &= 0 \\
  b_{21}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n &= 0 \\
  \vdots \\
  b_{p1}x_1 + b_{p2}x_2 + \ldots + b_{pn}x_n &= 0
\end{align*}
\]

Example: In the vector space \( \mathbb{R}^3 \), are the subspaces \( F \) and \( G \) defined by

\[
\begin{align*}
  x - y + z &= 0 \\
  x + y - z &= 0 \\
  x - 2y + 2z &= 0
\end{align*}
\]

respectively, relative to base \( B = \{(1,1,1),(0,1,-1),(-1,-1,0)\} \). Determine \( F \cap G \).

By the previous observation, \( F \cap G \) is defined relative to the base \( B = \{(1,1,1),(0,1,-1),(-1,-1,0)\} \) for

\[
\begin{align*}
  x - y + z &= 0 \\
  x + y - z &= 0 \\
  x - 2y + 2z &= 0 \\
  y + z &= 0
\end{align*}
\]

\[
\begin{bmatrix}
  1 & -1 & 1 & 0 \\
  1 & 1 & -1 & 0 \\
  1 & -2 & 2 & 0 \\
  0 & 1 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix} \leftrightarrow \begin{bmatrix}
  x = 0 \\
  y = 0 \\
  z = 0
\end{bmatrix}
\]
So, \( F \cap G = \{0(1,1,1)+0(0,1,-1)+0(-1,-1,0)\} = \{(0,0,0)\}, \) i.e., the null subspace.

\[ x + y - 2z = 0, \]

Example. \( P^2 \) in the vector space , are the vector subspaces \( F \), defined by

with a basic caution , and \( G = \langle x^2+x-1, x^2+2x+1 \rangle \). determine \( F \cap G \)

\[
G = \{ \alpha(x^2+x-1)+\beta(x^2+2x+1): \alpha, \beta \in \mathbb{R} \} = \{(\alpha+\beta)x^2+\alpha+2\beta+\alpha+\beta: \alpha, \beta \in \mathbb{R} \}
\]

This is, \( G =_B (\alpha+\beta, \alpha+2\beta, -\alpha+\beta) \), onde \( B = \{x^2, x, 1\} \) (standard basis \( P^2 \)).

So,

\[
F \cap G = \{(\alpha+\beta)x^2+(\alpha+2\beta)x-\alpha+\beta: 4\alpha+\beta=0 \} = \langle -3x^2-7x-5 \rangle
\]

Exemple. In the real vector space \( P^2 \), are the vector subspaces

\( F = \langle x^2-x+1, x^2+2x+2 \rangle \) e \( G = \langle 2x^2+x+1, -x^2-x+3 \rangle \).

Determine \( F \cap G \).

Now,

\[
F = \{ \alpha(x^2-x+1)+\beta(x^2+2x+2): \alpha, \beta \in \mathbb{R} \} = \{(\alpha+\beta)x^2+(-\alpha+2\beta)x+\alpha+2\beta: \alpha, \beta \in \mathbb{R} \}
\]

The actual values is determined to \( \alpha \) and \( \beta \), So that you vectors

\[ (\alpha+\beta)x^2+(-\alpha+2\beta)x+\alpha+2\beta \]

belonging to the vector subspace \( G \):

\[ (\alpha+\beta)x^2+(-\alpha+2\beta)x+\alpha+2\beta \]

\[ = \lambda_1(2x^2+x+1)+\lambda_2(-x^2-x+3) \]

\[
\begin{bmatrix}
2 & -1 \\
1 & -1 \\
1 & 3
\end{bmatrix}
\begin{bmatrix}
\alpha + \beta \\
-\alpha + 2\beta \\
\alpha + 2\beta
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
2\alpha - \beta \\
3\alpha - 3\beta \\
-20\alpha + 24\beta
\end{bmatrix}
\]

So, \(-20\alpha+24\beta=0 \Rightarrow \alpha=-8/5 \beta \ e \ \beta \in \mathbb{R}, \)

So, \( F \cap G = \langle -3/5 \beta x^2+18/5 \beta x+2/5: \beta \in \mathbb{R} \rangle = \langle -3x^2+18x+2 \rangle \).

Theorem (Subspaces Union). Let \( V \) a real vector space, and is \( F, G \subseteq V \). So \( F \cup G \) is a vector subspace of \( V \) if and only if, \( F \subseteq G \) ou \( G \subseteq F \).
Note (Important!!!). If F and G are vector subspace, of finite dimensions of a real vector space V, then \( F \subseteq G \) if and only if each of the generators F belong to G, that is linear combination of the generators G. If \( F \subseteq G \), then \( \text{dim}(F) \leq \text{dim}(G) \), in which case \( F + G = G \).

Example. In the real vector space \( P^2 \), are \( F = \langle x^2 - x \rangle \), \( G = \langle x - 1, 1 \rangle \). Check that \( F + G \) is a vector subspace of \( P^2 \).

However, because \( G \not\subseteq F \) then \( \text{dim}(G) = 2 > 1 = \text{dim}(F) \). \( F \subseteq G \) to each of the generators \( G \) is a linear combination of generators, i.e., the equation \( x^2 - x = \lambda_1(x - 1) + \lambda_2 \cdot 1 \) it’s possible.

But this equation is impossible because the lower degree polynomials or equal to a non degenerate polynomial of degree two. So \( F \not\subseteq G \). Thus, the \( F + G \) subset is not a vector subspace \( P^2 \).

Definition. Let \( V \) be an arbitrary real vector space, and \( F \) and \( G \) are subsets of \( V \). The sum of \( F \) and \( G \) is

\[
F + G = \{ f \vec{v} + g \vec{w} : f \vec{v} \in F \text{ and } g \vec{w} \in G \}.
\]

If one of the joint unit is, for example \( F = \{ f \vec{v} \} \), that is \( \{ f \vec{v} \} + G = f \vec{v} + G \).

Theorem. \( V \) is an arbitrary real vector space and are \( F, G \subseteq V \). So: \( F + G \subseteq V \).

1. If \( F = \langle \vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k \rangle \) and \( G = \langle \vec{g}_1, \vec{g}_2, \ldots, \vec{g}_p \rangle \), then \( F + G = \langle \vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k, \vec{g}_1, \vec{g}_2, \ldots, \vec{g}_p \rangle \).

Example. In the real vector space \( R^4 \), are the subspace \( F = \langle (1,0,0,0), (0,1,0,1) \rangle \) and \( G = \{ a(1,1,1,1) + b(-1,0,1,2) + c(1,1,-2,1) + d(0,0,1,1) : a - b + c = b - c + d = 0 \} \). Determine the size of \( F + G \).

Verify that \( F + G \) is an improper subspace of \( R^4 \), or not.

Now,

\[
\begin{align*}
\begin{cases}
  a - b + c = 0 \\
  b - c + d = 0
\end{cases} \iff \begin{cases}
  a = -d \\
  b = c - d
\end{cases}.
\end{align*}
\]

Soon,

\[
G = \{-d(1,1,1,1) + c-d(-1,0,1,2) + c(1,1,-2,1) + d(0,0,1,1) : c, d \in \mathbb{R} \}
\]

\[
= \{d(0,-1,-1,2) + c(0,1,-1,3) : c, d \in \mathbb{R} \}
\]

\[
= \{(0,-1,-1,2), (0,1,-1,3) \}
\]

Then, \( F + G = \langle (1,0,0,0), (0,1,0,1), (0,1,-1,2), (0,1,-1,3) \rangle \). Calculate a base \( F + G \), determining a maximal independent set of subsystem \( F + G \) generator.
As \( c(A) = 4 \), then \( \dim(F + G) = 4 \) and \( F + G = \mathbb{R}^4 \), or, \( F + G \) is an improper subspace of \( \mathbb{R}^4 \).

Theorem (dimensions). \( V \) is an arbitrary real vector space, and are \( F, G \leq V \). If \( F \) and \( G \) have finite dimensions, then

\[
\dim(F + G) = \dim(F) + \dim(G) - \dim(F \cap G).
\]

Example. Determine \( F \cap G \), where \( F \) and \( G \) are the vector subspaces of the preceding example.

\( \dim(F) = 2 \), \( \dim(G) = 2 \) e \( F + G = \mathbb{R}^4 \). Then the theorem of dimensions, \( 4 = 4 + \dim(F \cap G) \iff \dim(F \cap G) = 0 \iff F \cap G = \{(0,0,0,0)\} \).

Definition (direct sum). \( V \) is an arbitrary real vector space, and are \( F, G \leq V \). It is said that \( S = F + G \) is a direct sum and is denoted by \( S = F \oplus G \), if \( F \cap G = \{0 \vec{\cdot}\} \).

Example. In the previous example, \( S = F + G = F \oplus G \).

Theorem. \( V \) an arbitrary real vector space, and are \( F, G \leq V \). If \( B_F \) is a base of \( F \), \( B_G \) is a base of \( G \) and \( S = F + G = F \oplus G \), then \( B = B_F \cup B_G \) is a base of \( S = F \oplus G \).

Definition. \( V \) is an arbitrary real vector space, and are \( F_1, F_2, \ldots, F_k \leq V \). It is said that the sum \( S = F_1 + F_2 + \cdots + F_k \) is a direct sum, and writes \( S = F_1 \oplus F_2 \oplus \cdots \oplus F_k \), for all \( i \in \{1, 2, \ldots, k\} \) there has \( F_i \cap (\sum_{i \neq j=1}^{k} F_j) = \{0 \vec{\cdot}\} \).

Example. In the real vector space \( P^3 \), are \( F = \langle x^3, x^2 \rangle \), \( G = \langle x^3 + x \rangle \) and \( H = \langle x^3 + 1, 2x^2 + 3x \rangle \). Check that \( S = F + G + H \) is a direct sum.

\( S = F + G + H = F \oplus G \oplus H \) is \( \text{Fn}(G + H) = \text{Gn}(F + H) = \text{Hn}(F + G) = \{0 \vec{\cdot}\} \). Now, \( G + H = \langle x^3 + x, x^2 + 1, 2x^2 + 3x \rangle \), and how these three vectors are linearly independent, then \( \dim(G + H) = 3 \).

The theorem of dimension,

\[
\dim(F + (G + H)) = \dim(F) + \dim(G + H) - \dim(\text{Fn}(G + H)).
\]

As \( F + (G + H) \leq P^3 \), and therefore \( \dim(F + (G + H)) \leq \dim(P^3) = 4 \), then it cannot be that \( \dim(\text{Fn}(G + H)) = 0 \), it would be

\[
\dim(F + (G + H)) = \dim(F) + \dim(G + H) = 5 > \dim(P^3) = 4,
\]

i.e., \( \text{Fn}(G + H) \neq \{0 \vec{\cdot}\} \), therefore \( S = F + G + H \) is not a direct sum.

Definition (Complementary subspace). Let \( V \) be an arbitrary real vector space, and let \( F \leq V \). \( F \) is called a complement of a vector subspace \( F' \) of \( V \) such that \( F \oplus F' = V \).

Notice. If \( F = V \), then \( F' = \{0 \vec{\cdot}\} \); and \( F = \{0 \vec{\cdot}\} \), then \( F' = V \).
Theorem. Let $V$ be a real vector subspace of finite dimension, and let $F \subseteq V$. Then there is a complementary subspace of $F$.

Note (Important!!!). Chosen arbitrary base $P^4$ (preferably the canonical basis) and determines if a new base $P^4$ that includes $F$ of based vector, applying Steinitz Theorem (for example). Thus, the base $F'$ is composed of the base vector is determined not forming part of the base $F$.

Example. In the real vector space $P^4$, let $F=\langle 1, x+1, x^2+1 \rangle$. Determine $F'$.

In this example, let $B=\{x^4, x^3, x^2, x, 1\}=\{e_1, e_2, e_3, e_4, e_5\}$ canonical basis $P^4$. It has:

\[ f_1=1=e_5, \quad f_2=e_4+e_5, \quad f_3=e_3+e_5. \]

Soon:

\[ B_1=\{e_1, e_2, e_3, f_1\}, \quad f_2=e_4+f_1, \quad f_3=e_3+f_1. \]

\[ B_2=\{e_1, e_2, e_3, f_2\}, \quad f_3=e_3+f_1. \]

\[ B_3=\{e_1, e_2, f_3\}. \]

Then $F'=\langle e_1, e_2 \rangle = \langle x^4, x^3 \rangle$.

Exercises.

1. In the real vector $P^3$, let the vector subspace $F=\langle x^3+x^2, x^3-1 \rangle$ and $G$ represented by the system $a-b+c-d=a+c+2d=-b+c+d=0$, for the base $B=\{x^3-x^2, x^2-2x, x-1, -x^3+x+2\}$.

   Determine $F+G$.

   Represent $F+G$ by the mean of a system of linear homogeneous equation, relative to the base $B$.

   Determine $F \cap G$.

2. Verify that $x=(1, 2, 0, -1)$ belong to $F$.

   Construct a vector subspace $G$ of $R^4$, two dimension, such that $G \neq F$ e $G \cap F=\langle x \rangle$.

   State what is the dimension of $G+F$.

   Construct a complementary subspace of $F$ in $R^4$.

3. In the real vector space $R^3$, let $F=\{(x, y, z): 3x+y=x-z=0\}$ and $G=\{(x, y, z): kx+2y-z=0\}$, with $k \in \mathbb{R}$.

   Determine the actual value of $k$ for which $F \cup G$ is a vector subspace of $R^4$.

4. In the real vector space $R^4$, let the the vector subspace be $F=\{(1, 0, 1, 0), (0, 1, -1, 0), (1, 1, 1, 1)\}$ and $G=\{(x, y, z, w): x+y+w=x+2z+w=0\}$.

   Determine $F \cap G$ and $F+G$.

   Determine whether $F \cup G \subseteq R^4$. 

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5. In the real vector space $R^3$, let the vectors subspace be $F=\langle (1,0,0) \rangle$, $G=\{(x,y,z):x+y=0\}$ and $H=\{(x,y,z):2x-y=x+z=0\}$. Verify that:

\[ F+G=F \oplus G. \]
\[ F+G+H=F \oplus G \oplus H. \]

6. In the real vector space $P^3$, it is defined by $a+b-c+d=a+b+c-d=0$, relative to the standard basis.

Determine the base of $F$.

Determine the base of $P^3$ that includes the vectors of base of $F$, determine in the previous paragraph.

Determine a complementary subspace of $F$.

And $H=\langle x^3+x^2+kx+1,kx^3+x^2+kx+1 \rangle$, discuss when the actual parameter $k$ a dimension of $F \cap G$.

7. In the real vector space $R^4$, let the vectors space be $F=\{(x,y,z,w):x+y+z+w=0\}$ and $G=\{(x,y,z,w):x=y=z=0\}$.

Verify that $F \oplus G=R^4$.

Indicate other two vector subspaces $U$ and $V$ of $R^4$, different from $F$ and $G$, such that $U \oplus V=R^4$.

**Conclusion**

Knowledge of vectors of properties in a real vector space is fundamental.

The concept of linear combination, the real vector space, giving rise to other fundamental concepts such as linear dependence and independence generator and, consequently, base and dimension.

**Assessment**

Show which of the following subsets are vector subspace of the vector spaces indicated, indicating for each of a basis:

$S=\{(x,y,z,w):x=0=y+z\}$ in $R^4$.

$S=\{ax^3+bx^2+cx+d:a \text{ is irrational}\}$ in $P^3$.

$S=\{(x,y,z,w):x+y=2z-w=0\}$ in $R^4$.

$S=\{p(x):p(0)=0\}$ in $P^5$.

$S=\{ax^2+bx+c:b \geq 0\}$ in $P^2$.

$S=\{(x,y,z,w):xy=0\}$ in $R^4$.

$S=\{(x,y,z,w):|x| \geq 2\}$ in $R^4$.

$S=\{ax^2+(a-b)x+b:a,b \in \mathbb{R}\}$ in $P^2$. 
Find a basis for each of the vector subspaces of vector spaces indicated:

\[ S = \{ (x,y,z,w) : x + 2y - z = x + y + 2w = y - z + w = 0 \} \] in \( \mathbb{R}^4 \).

\[ S = \{ a(x^2 - x) + b(x^2 + 1) + c(x + 2) : a - b + 2c = 0 \} \] in \( \mathbb{P}^2 \).

Define, by means of linear homogeneous equation, each of the vector subspaces of the real vector space \( \mathbb{P}^3 \) in relation to the basis given:

\[ S = \langle x^2 - 1 \rangle \] with respect to the basis \( B = \{ x^2 - x, x^2 + 1, x + 2 \} \).

\[ S = \langle x^2 - 1 \rangle \] with respect to the canonical basis.

\[ S = \langle x^2 - x, xx + 1 \rangle \] with respect to the basis \( B = \{ x^2 - x + 1, x - 1, -x + 2 \} \).

\[ S = \langle x^2 - x, xx + 1 \rangle \] with respect to the canonical basis.

Determine the vector subspace of \( \mathbb{P}^3 \), defined by the system

\[ \begin{align*}
&x - y + z + 2w = 0 \\
&x - 2y + z + w = 0
\end{align*} \]

with respect to the basis \( B = \{ x^3 - x^2, x^2 - x, x - 1, x^3 \} \).

Determine the vector subspace of \( \mathbb{R}^4 \), defined by the system

\[ \begin{align*}
&x - y + z + 2w = 0 \\
&x - 2y + z + w = 0
\end{align*} \]

Determine a basis and a dimension of each of the vector subspaces of vector spaces given:

\[ S = \{ a(x^3 - x^2) + b(x^2 - x) + c(x - 1) + dx^3 : a - b = a + b - c + 2d = 0 \} \] in \( \mathbb{P}^3 \).

\[ S = \{ x(1,1,1) + y(0,-2,1) + z(-1,-3,1) : x - y = y - z = 0 \} \] in \( \mathbb{R}^3 \).

\[ S = \{ (3x + y, y + z, z, 0) : x, y, z \in \mathbb{R} \} \] in \( \mathbb{R}^4 \).

\[ S = \{ ax^3 + (-a + b)x^2 + (-b + c + 2d)x - c - 2d : a, b, c, d \in \mathbb{R} \} \] in \( \mathbb{P}^3 \).

**Activity 2.2 - Real vector space with inner product**

**Introduction**

It introduces the concept of matrix change of basis, as a tool that allows relating different bases of a given real vector space.

It introduces also the domestic product in the vector space, highlighting the spaces of finite dimensions, in order to give certain characteristics to vectors.

**Activity Details**

**Matrix Base Change**

Definition (Matrix base change). Let \( V \) be a real vector space \( n \in \mathbb{N} \) dimension. Let \( B = \{ v_1, v_2, \ldots, v_n \} \) be a basis of \( V \) (Old base), and let \( B' = \{ v'_1, v'_2, \ldots, v'_n \} \) another base of \( V \) (New base). As \( B' \subset V \), then each vector of \( B' \) is a linear combination of vectors of \( B \):

\[ v'_1 = \sum_{i=1}^{n} a_i v_i \]
The matrix

\[
M(B \rightarrow B') = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

is called a matrix base change, the old base \( B \) to the new base \( B' \).

Note that the first column of matrix \( M(B \rightarrow B') \) consist of the vector components \( v' \_1 \) the base \( B \), the second column, the components of the vector \( v' \_2 \) the base \( B \), and so on.

**Theorem.** Every change of basis matrix is invertible (or regular). An invertible matrix may be regarded as a matrix base change, that is, given an invertible matrix \( Q \), of order \( n \), it is impossible to have two bases \( B \) and \( B' \) of certain vector space \( V \), of dimension \( n \), such that \( Q=M(B \rightarrow B') \). Let \( P=M(B \rightarrow B') \) (change of basis matrix, \( B \) to \( B' \)), then \( P^{-1}=M(B' \rightarrow B) \) is change of basis matrix, \( B' \) to \( B \).

Note that the first column of matrix \( P^{-1} \) is composed of components of the vector \( v' \_1 \) in the base \( B' \), the second column, the components of vector \( v' \_2 \) in the base \( B' \), and so on.

**Theorem.** Let \( V \) be a real vector space of dimension \( n \), and let \( B=\{v' \_1,v' \_2,\ldots,v' \_n\} \) and \( B' '=\{v \_1,v \_2,\ldots,v \_n\} \) two bases of \( V \). Let \( v \) be an arbitrary vector \( V \). Then

\[
v' = \lambda_1 v' \_1 + \lambda_2 v' \_2 + \cdots + \lambda_n v' \_n \quad \text{and} \quad v = [\lambda^\top] \_1 v' \_1 + [\lambda^\top] \_2 v' \_2 + \cdots + [\lambda^\top] \_n v' \_n.
\]

If \( P=M(B \rightarrow B') \), then

\[
P [\begin{bmatrix}
  \lambda'_1 \\
  \lambda'_2 \\
  \vdots \\
  \lambda'_n
\end{bmatrix}] = [\begin{bmatrix}
  \lambda_1 \\
  \lambda_2 \\
  \vdots \\
  \lambda_n
\end{bmatrix}].
\]

**Example.** In the vector space \( P^2 \), it is considered the bases

\( B=\{x^2+1,x-1,x^2+x+1\} \) and \( B'=\{x^2,x,1\} \).

a. Determine a matrix \( P=M(B' \rightarrow B) \).

b. Show that \( PX^t \), where \( X=[\lambda_1 \lambda_2 \lambda_3] \) and \( ax^2+bx+c \equiv_B (\lambda_1 \lambda_2 \lambda_3) \), has as input the components of \( ax^2+bx+c \) in the base \( B' \).

c. Determine \( Q=M(B \rightarrow B')=P^{-1} \)
A) The matrix base change from $B'$ to $B$ is $P = M(B' \rightarrow B)$, i.e. the first column $P$ is composed by vector $x^2+1$ components on the base $B'$, the second column consist of $P$ by vector $x-1$ components in the base $B'$ and the last column is composed of $P$ component of the vector $x^2+x+1$ in the base $B'$. As $B'$ standard base is $P^2$, then: $x^2+1 \equiv_{B'} (1,0,1)$, $x-1 \equiv_{B'} (0,1,-1)$ and $x^2+x+1 \equiv_{B'} (1,1,1)$. Soon

$$P = M(B' \rightarrow B) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$ 

B). However $ax^2+bx+c \equiv_{B} (2a-b-c,a-c,-a+b+c)$, for all $a,b,c \in \mathbb{R}$. Then

Now $ax^2+bx+c \equiv_{B'} (a,b,c)$

C). The array changing base $B$ to $B'$ is

$$Q = P^{-1} = M(B \rightarrow B') = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$ 

Note that:

$x^2 \equiv_{B} (2,1,-1)$

$x \equiv_{B} (-1,0,1)$

$1 \equiv_{B} (-1,-1,1)$

i.e. the first column of the matrix $Q$ base change vector $x^2$ is composed of the components in the base $B$, the second column, the components of the vector $x$ at the base $B$ and the third column, the first vector component on the base $B$. Exemple. Let

$$Q = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ -1 & -1 & -1 \end{bmatrix}$$

be an invertible matrix.

a) Determine two bases $B$ and $B'$ of $\mathbb{R}^3$, such that $Q = M(B' \rightarrow B)$.

b) Knowing that $Q = M(B \rightarrow B')$ and $B' = \{ (1,0,-1),(1,1,2),(2,1,2) \}$, determine $B$.

A). Let $B = \{ v_1,v_2,v_3 \}$ and $B' = \{ v_1',v_2',v_3' \}$ are two bases of $\mathbb{R}^3$. So
\[
\begin{align*}
\bar{v}_1 &= \bar{v}'_1 + \bar{v}'_2 - \bar{v}'_3 \\
\bar{v}_2 &= -\bar{v}'_1 + \bar{v}'_2 - \bar{v}'_3 \\
\bar{v}_3 &= \bar{v}'_1 - 2\bar{v}'_2 + \bar{v}'_3.
\end{align*}
\]

For \(B' = \{(1,0,-1),(1,-1,-4),(1,-4,-11)\}\), then \(B = \{(1,3,6),(-1,3,8),(-2,6,18)\}\).

B). \(Q = M(B \rightarrow B')\) and matrix \(B'\) is given, then it will be express each of the base vector \(B\) as a function of the vectors of base \(B'\):

\[
\begin{align*}
\bar{v}'_1 &= \frac{1}{2} \bar{v}'_1 - \frac{1}{2} \bar{v}'_2 \\
\bar{v}'_2 &= \frac{1}{3} \bar{v}'_1 - \frac{1}{3} \bar{v}'_3 \\
\bar{v}'_3 &= -\frac{1}{6} \bar{v}'_1 - \frac{1}{2} \bar{v}'_2 - \frac{1}{3} \bar{v}'_3.
\end{align*}
\]

As \(B' = (1,0,-1),(1,1,2),(2,1,2)\), then making substitutions in the above equation, it has:

\(B = \{v_1, v_2, v_3\} = \{(0,-1/2,-3/2),(-1/3,-1/3,-1),(-4/3,-5/6,-3/2)\}\).

Exercices.

1. In the real vector space of real \(P^2\) there are bases

\(B = \{x^2-x+1,-x+1,-x^2-x+2\}\) and \(B' = \{x^2-x,-x^2+2x+1,-x^2+x+1\}\).

   a) Determine \(P = M(B \rightarrow B')\).
   b) Show that \(Q = P^{-1} = M(B' \rightarrow B)\).
   c) Knowing that the matrix

\[
A = M(C \rightarrow B) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix},
\]

determine the basis \(C\).

2. Justify that the matrix

\[
P = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 3 & 4 \\ -1 & -2 & -2 \end{bmatrix}
\]
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is a change of basis matrix from a real vector space of dimension 3.

a) Determine two bases, B and B’, of R^3, such that P=M(B→B’).

b) Determine the components (α’,β’,γ’) of the vector (a,b,c), with a,b,c∈R, the base B’, determine above.

c) Show that the components (a,b,c), where a,b,c∈R, based on B, given above, is (α,β,γ),

\[
\begin{bmatrix}
\alpha \\
\beta \\
\gamma \\
\end{bmatrix}
= P
\begin{bmatrix}
\alpha' \\
\beta' \\
\gamma' \\
\end{bmatrix}.
\]

3. In the real vector space P^3, let the matrix

\[
P = M(B \rightarrow C) = \begin{bmatrix}
0 & 1 & 0 & 2 \\
-1 & 1 & 0 & -1 \\
0 & 0 & 2 & 2 \\
0 & -1 & 1 & 0 \\
\end{bmatrix},
\]

where B={x^3-x^2-x+1,x^2+1,-x^2-x-1,-2x^2+x-3}. Knowing the vector p(x)$_\equiv$ C (-1,1,2,-1), determine the vector p(x).

Real Vector Space with Inner product

Definition (Inner Product). Let V be a real vector space. An inner product (or scalar) V is a function that assigns to each ordered pair of vectors, u $\cdot$ and v $\cdot$ in V, a real scalar, denoted by u $\cdot$ v $\cdot$, which satisfied the following properties:

1. $\forall u \cdot, v \cdot, w \cdot \in V$, we have $(u \cdot + v \cdot) \cdot w \cdot = u \cdot \cdot w \cdot + v \cdot \cdot w \cdot$;
2. $\forall u \cdot, v \cdot, \lambda \in R$, we have $(\lambda u \cdot) \cdot v \cdot = \lambda (u \cdot \cdot v \cdot)$;
3. $\forall u \cdot, v \cdot \in V$, we have $u \cdot \cdot v \cdot$ = $v \cdot \cdot u \cdot$;
4. $\forall u \cdot \in V$ and $u \cdot \neq 0 \cdot$, we have $u \cdot \cdot u \cdot > 0$.

Theorem. In a real vector space V inner product, the following properties hold:

1. $\forall u \cdot, v \cdot, w \cdot \in V$, we have $u \cdot (w \cdot + v \cdot) = u \cdot \cdot w \cdot + u \cdot \cdot v \cdot$;
2. $\forall u \cdot, v \cdot \in V$, $\forall \lambda \in R$, we have $(\lambda v \cdot) \cdot u \cdot = \lambda (u \cdot \cdot v \cdot)$.
3. $u \cdot \cdot u \cdot = 0$ if and only if, $u \cdot \cdot u \cdot = 0$.

Exemple (Standard Inner Product in R^n). In the real vector space R^n, we define an inner product, called standard inner product as follow:

$\forall u \cdot = (u_1, u_2, u_3, \ldots, u_n), v \cdot = (v_1, v_2, v_3, \ldots, v_n)$, tem-se
\[ u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 + \cdots + u_n v_n. \]

Proof:

\[ \forall u = (u_1, u_2, u_3, \ldots, u_n), \; v = (v_1, v_2, v_3, \ldots, v_n), \; w = (w_1, w_2, w_3, \ldots, w_n) \in \mathbb{R}, \; \lambda \in \mathbb{R}: \]

Property (1),

\[(u + v) \cdot w = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots, u_n + v_n) \cdot w = (u_1 w_1 + u_2 w_2 + u_3 w_3 + \cdots + u_n w_n) + (v_1 w_1 + v_2 w_2 + v_3 w_3 + \cdots + v_n w_n) = u \cdot w + v \cdot w \]

Property (2),

\[(\lambda u) \cdot v = (\lambda u_1, \lambda u_2, \lambda u_3, \ldots, \lambda u_n) \cdot v = \lambda (u_1 v_1 + u_2 v_2 + u_3 v_3 + \cdots + u_n v_n) = \lambda (u \cdot v) \]

Property (3),

\[ u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 + \cdots + u_n v_n \]

Property (4). If \( u \neq 0 \), then there exists \( i \in \{1, 2, 3, \ldots, n\} \) such that \( u_i \neq 0 \), therefore

\[ u \cdot u = [u_1]^2 + [u_2]^2 + [u_3]^2 + \cdots + [u_n]^2 > 0. \]

Example (Inner Product in Polynomial Vector Space). In vector spaces \( P \) and \( P^n \), with \( n \in \mathbb{N}_0 \), define the following inner product:

\[ \forall a, b \in \mathbb{R}, \forall u = p(x), v = q(x) \in P, \; u \cdot v = \int_a^b p(t) q(t) dt. \]

Remark. A proof is made easily from the full set of properties.

Example. In the vector space \( P \), determine \( \| p(x) = -x + 1 \| \), for the following inner product

\[ \forall a, b \in \mathbb{R}, \forall u = p(x), v = q(x) \in P, \; u \cdot v = \int_{-1}^1 p(t) q(t) dt. \]

Example. In the vector space \( P \), determine \( \| p(x) = -x + 1 \| \), for the following inner product

\[ \| p(x) = -x + 1 \| = \sqrt{(p(x) \cdot p(x))} = \sqrt{\int_{-1}^1 p(t) q(t) dt}. \]

Exemple (Inner Product in Polynomial Vector Space). In vector spaces \( P \) and \( P^n \), with \( n \in \mathbb{N}_0 \), define the following inner product:

\[ \forall a, b \in \mathbb{R}, \forall u = p(x), v = q(x) \in P, \; u \cdot v = \int_{-1}^1 p(t) q(t) dt. \]

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\[ \| p(x) = -x + 1 \| = \sqrt{(p(x) \cdot p(x))} = \sqrt{\int_{-1}^1 p(t) q(t) dt}. \]
Exemple (Other inner products). Inner products are also the following:

1. \( \forall \mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in \mathbb{R}^2, \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_1 + u_1 v_2 + 2u_2 v_2; \)

2. \( \forall \mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3, \)
\[ \mathbf{u} \cdot \mathbf{v} = 5u_1 v_1 + 2u_2 v_1 - u_3 v_1 + 2u_1 v_2 + 2u_2 v_2 - 3u_3 v_2 - u_1 v_3 - 3u_2 v_3 + 12u_3 v_3. \]

Definition (Norm). Let \( V \) be a real vector space with inner product. For each \( \mathbf{x} \in V \), called norm of \( \mathbf{x} \) the real number not negative \( \| \mathbf{x} \| = \sqrt{\mathbf{x} \cdot \mathbf{x}} \).

Exemple. In the real vector \( \mathbb{R}^3 \), determine the norm of vector \( \mathbf{x} = (1, -1, 2) \), compare:

(a) Standard inner Product;

(b) \( \forall \mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3, \)
\[ \mathbf{u} \cdot \mathbf{v} = 5u_1 v_1 + 2u_2 v_1 - u_3 v_1 + 2u_1 v_2 + 2u_2 v_2 - 3u_3 v_2 - u_1 v_3 - 3u_2 v_3 + 12u_3 v_3. \]

A). \( \| \mathbf{x} \| = \sqrt{1 + 1 + 4} = \sqrt{6}. \)

B). \( \| \mathbf{x} \| = \sqrt{5 - 2 - 2 + 2 + 6 - 2 + 6 + 48} = \sqrt{59}. \)

Theorem (The Norm Properties). Let \( V \) be a real vector space with an inner product. For any \( \mathbf{x}, \mathbf{y} \in V, \forall \lambda \in \mathbb{R} \), it has:

1. \( \| \mathbf{x} \| = 0, \) if and only if, \( \mathbf{x} = 0. \)

2. \( \| \lambda \mathbf{x} \| = |\lambda| \| \mathbf{x} \|. \)

3. \( |\mathbf{x} \cdot \mathbf{y}| \leq \| \mathbf{x} \| \| \mathbf{y} \| \) (Cauchy-Schwarz Inequality). Dá-se a igualdade se, e somente se, os vectores \( \mathbf{x} \) e \( \mathbf{y} \) são proporcionais.

4. \( \| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \| \) (Triangular Inequality). It gives equal if and only if one of the vector can be obtained at the expense of the other , by multiplication of a non-negative real scalar.

Definition (Angle). Let \( V \) be a real vector space with an inner product. Let \( \mathbf{x}, \mathbf{y} \in V, \mathbf{x}, \mathbf{y} \neq 0, \) so called the angle \( \mathbf{x} \) and \( \mathbf{y} \) (or between \( \mathbf{x} \) and \( \mathbf{y} \)), and is represented \( \angle (\mathbf{x}, \mathbf{y}) \), the value \( \theta \in [0, \pi] \) such that
\[ \cos(\theta) = \frac{(\mathbf{x} \cdot \mathbf{y})}{(\| \mathbf{x} \| \| \mathbf{y} \|)}, \]
\[ \theta = \angle (\mathbf{x}, \mathbf{y}) = \arccos\left(\frac{(\mathbf{x} \cdot \mathbf{y})}{(\| \mathbf{x} \| \| \mathbf{y} \|)}\right). \]

Exemple. In the real vector space \( \mathbb{R}^3 \), there are \( \mathbf{x} = (1, -1, 1) \) e \( \mathbf{y} = (-1, 2, -2) \). Determine \( \angle (\mathbf{x}, \mathbf{y}) \), considering:

a. The standard inner product.

b. \( \forall \mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3, \)
\[ \mathbf{u} \cdot \mathbf{v} = 5u_1 v_1 + 2u_2 v_1 - u_3 v_1 + 2u_1 v_2 + 2u_2 v_2 - 3u_3 v_2 - u_1 v_3 - 3u_2 v_3 + 12u_3 v_3. \]
A). \( \vec{x} \cdot \vec{y} = -1 - 2 - 2 = -5 \); \( \| \vec{x} \| = \sqrt{3} \); \( \| \vec{y} \| = \sqrt{9} = 3 \). Then:
\[ \angle(\vec{x}, \vec{y}) = \arccos\left(\frac{-5}{3\sqrt{3}}\right) \approx 164^\circ. \]

B). \( \vec{x} \cdot \vec{y} = -5 + 2 + 4 - 4 - 6 - 24 = -36 \); \( \| \vec{x} \| = \sqrt{19} \); \( \| \vec{y} \| = \sqrt{73} \). Then:
\[ \angle(\vec{x}, \vec{y}) = \arccos\left(\frac{-36}{\sqrt{1387}}\right) \approx 165^\circ. \]

Theorem (Angles Properties). Let \( V \) be a real vector space, with a fixed inner product. For any vector \( \vec{x}, \vec{y} \in V \) and \( \vec{x} \neq 0, \vec{y} \neq 0 \), it has:

1. \( \angle(\vec{x}, \vec{x}) = 0 \);
2. \( \angle(\vec{x}, \vec{y}) = \angle(\vec{y}, \vec{x}) \);
3. \( \angle(\vec{x}, \vec{y}) = \angle(\lambda \vec{x}, \beta \vec{y}) \), where \( \lambda \) and \( \beta \) are scalar having the same sign;
4. \( \angle(\vec{x}, \vec{y}) = \pi - \angle(\lambda \vec{x}, \beta \vec{y}) \), where \( \lambda \) and \( \beta \) are scalar with opposite signs.

Definition (Orthogonality). Let \( V \) be a real vector space with inner product. It is said that a vector \( \vec{x} \) a vector orthogonal to \( \vec{y} \), and is represented by \( \vec{x} \perp \vec{y} \), if \( \vec{x} \cdot \vec{y} = 0 \).

Theorem (Orthogonality Properties). Let \( V \) be a real vector space with an inner product. For any vector \( \vec{x}, \vec{y} \in V \), it has:

1. If \( \vec{x} \perp \vec{y} \), then \( \vec{y} \perp \vec{x} \).
2. \( 0 \perp \vec{x} \).
3. \( \vec{x} \perp \vec{x} \) if and only if, \( \vec{x} = 0 \).
4. If \( \vec{x} \perp \vec{y} \), then if \( \vec{x} \perp \lambda \vec{y} \), \( \forall \lambda \in \mathbb{R} \).

Definition (Orthogonal and Orthonormal system of Vectors). Let \( V \) be a real vector space with inner product. The vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in V \) form an Orthogonal System if each of them is orthogonal to each other, i.e., \( \forall i \in \{1,2,\ldots,k\} \) \( \vec{v}_i \cdot \vec{v}_j = 0 \), with \( i \neq j \). In addition, \( \forall i \in \{1,2,\ldots,k\} \) \( \| \vec{v}_i \| = 1 \), we say that \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) form an Orthonormal System, that is, for any \( i,j = 1,2,\ldots,k \), we have:
\[ \vec{v}_i \cdot \vec{v}_j = \begin{cases} 0, & \text{se } i \neq j \text{ se } i = j \end{cases} \]

Note. In a real vector space \( V \), having an internal product such that \( \vec{x} \in V \) and \( \| \vec{x} \| = 1 \), we say that \( \vec{x} \) is a Unit Vector (or normalized).

Definition (Inverse of a vector). Let \( V \) be a real vector space with inner product. Inverse of a vector \( \vec{u} \in V \) is the vector \( \frac{1}{\| \vec{u} \|} \vec{u} = \frac{\vec{u}}{\| \vec{u} \|} \).

Note. Inverse of a vector \( \vec{u} \in V \) is unit (or normalized). Just look at that:
\[ \| \vec{u} \| = \| \vec{u} \| = |1/\| \vec{u} \| | \| \vec{u} \| = 1/\| \vec{u} \| \| \vec{u} \| = (\| \vec{u} \|)/\| \vec{u} \| = 1. \]
Unit 2. Real Vector Space

Exemple. In a real vector space $R^3$, there are vectors $\vec{v_1}=(1,1,1)$, $\vec{v_2}=(-1,1,0)$ and $\vec{v_3}=(-1,-1,2)$.

a. Show that $S=\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is orthogonal, to the standard inner product

b. Get an orthonormal system from $S$.

A). $\vec{v_1} \cdot \vec{v_2}=-1+1+0=0; \vec{v_1} \cdot \vec{v_3}=-1-1+2=0; \vec{v_2} \cdot \vec{v_3}=1-1+0=0$. Therefore, $S=\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is orthogonal.

B). $\|\vec{v_1}\|=\sqrt{1+1+1}=\sqrt{3}; \|\vec{v_2}\|=\sqrt{1+1}=\sqrt{2}; \|\vec{v_3}\|=\sqrt{1+1+4}=\sqrt{6}$. Find the inversers of $\vec{v_1}, \vec{v_2}, \vec{v_3}$:

$u_1=\vec{v_1}/(\|\vec{v_1}\|)=(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}); u_2=\vec{v_2}/(\|\vec{v_2}\|)=((-1)/\sqrt{2},1/\sqrt{2},0); u_3=\vec{v_3}/(\|\vec{v_3}\|)=((-1)/\sqrt{6},(-1)/\sqrt{6},2/\sqrt{6})$.

Then, the vectors $u_1, u_2, u_3$ form a orthonormal system.

Note. The process of conversion of an orthogonal system $S$ in an orthonormal system, the inverse consist in determining for each of the vector $S$.

Theorem. All orthogonal system formed by non-zero vector is linearly independent. In particular, all orthonormal system is linearly independent.

Definition (Orthogonal Projection). Let $V$ be a real vector space with inner product. Let $\vec{v}, \vec{w} \in V$ and $\vec{w} \neq 0$. be an orthgonormal projection of $\vec{v}$ on $\vec{w}$ is the vector $[\text{proj}]_{\vec{w}} \vec{v}=(\vec{v} \cdot \vec{w})/\|\vec{w}\|^2 \vec{w}$.

In the plane and the usual space ($R^2$ and $R^3$), an orthogonal projection of $\vec{v}$ on $\vec{w}$ can be represented:

$$\vec{v} = \vec{v_{\text{proj}}} + \vec{v_{\text{ orth}}}. $$

Theorem. Let $V$ be a real vector space with an inner product. Let $\vec{w} \in V$ and $\vec{w} \neq 0$. Then, $\forall \vec{v}, \vec{v} - [\text{proj}]_{\vec{w}} \vec{v}$ is orthogonal to $\vec{w}$, or,

$$(\vec{v} - [\text{proj}]_{\vec{w}} \vec{v}) \cdot \vec{w} = 0.$$
Note (Very important!!!). $\vec{v} = \lambda_1 \vec{w}_1 + \lambda_2 \vec{w}_2 + \cdots + \lambda_k \vec{w}_k \in \langle X \rangle$ are referred to the above theorem such that $\vec{v} = \lambda_1 \vec{w}_1 + \lambda_2 \vec{w}_2 + \cdots + \lambda_k \vec{w}_k$.

Then, if the subspace $(X)$ has finite size, the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \in \langle X \rangle$ are vectors of a given base of $(X)$, such that $\vec{v}$ is their linear combination.

Example. In the real vector space $\mathbb{R}^3$, with an standard inner product, let $B = \{(1,1,1),(-1,1,0),(-1,-1,2)\}$ on its base. For the example above, $B$ is composed of non-zero orthogonal vectors.

Then, for any vector $\vec{v} = (a,b,c) \in \mathbb{R}^3$, there are real scalar (single) $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that:

$$\vec{v} = \lambda_1 (1,1,1) + \lambda_2 (-1,1,0) + \lambda_3 (-1,-1,2).$$

Therefore, by the above theorem, we have:

$$\lambda_i = (\vec{v} \cdot \vec{w}_i) / ||\vec{w}_i||^2, i = 1,2,3,$$

and $\vec{w}_1 = (1,1,1)$; $\vec{w}_2 = (-1,1,0)$ and $\vec{w}_3 = (-1,-1,2)$.

However, $||\vec{w}_1|| = \sqrt{3}$; $||\vec{w}_2|| = \sqrt{2}$ e $||\vec{w}_3|| = \sqrt{6}$. Then,

$$\lambda_1 = (\vec{v} \cdot \vec{w}_1) / ||\vec{w}_1||^2 = (a+b+c)/3,$$

$$\lambda_2 = (\vec{v} \cdot \vec{w}_2) / ||\vec{w}_2||^2 = (-a+b)/2,$$

$$\lambda_3 = (\vec{v} \cdot \vec{w}_3) / ||\vec{w}_3||^2 = (-a-b+2c)/6 = (-a-b)/6 + c/3.$$

Pythagorean Theorem. Let $V$ be a real vector space with an inner product. Let $\vec{u}, \vec{v} \in V$. If $\vec{u} \perp \vec{v}$, then $||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2$.

Exercises

1. Verify whether each the following indicates an inner product in vector space:
   a. $\forall \vec{u} = (u_1,u_2), \vec{v} = (v_1,v_2) \in \mathbb{R}^2, \vec{u} \cdot \vec{v} = 3u_1 v_1 + u_2 v_2 + 2u_2 v_2$.
   b. $\forall \vec{u} = (u_1,u_2), \vec{v} = (v_1,v_2) \in \mathbb{R}^2, \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_1 v_1 + u_1 v_2 + u_2 v_2$.
   c. $\forall \vec{u} = (u_1,u_2,u_3), \vec{v} = (v_1,v_2,v_3) \in \mathbb{R}^2, 2u_1 v_1 + 3u_3 v_3 + 2u_1 v_2 + 2u_2 v_3 + 3u_1 v_3 + u_3 v_3$.
   d. $\forall \vec{u} = (u_1,u_2,u_3), \vec{v} = (v_1,v_2,v_3) \in \mathbb{R}^2, u \cdot \vec{v} = 2u_1 v_1 + u_2 v_2 + u_3 v_3 + u_1 v_1 + u_1 v_2 + 3u_2 v_2 - u_2 v_3 + u_1 v_3 + 3u_2 v_3 + 2u_3 v_3$.

2. Consider the following inner product in real vector space $\mathbb{R}^3$:

$\forall \vec{u} = (u_1,u_2,u_3), \vec{v} = (v_1,v_2,v_3),$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3,$$

$$\vec{u} \cdot \vec{v} = 2u_1 v_1 + u_3 v_3 + 2u_2 v_2 + u_1 v_3 + u_2 v_3 + 2u_3 v_3.$$

$\vec{u} \cdot \vec{v} = 3u_1 v_1 + u_2 v_1 + u_1 v_2 + 2u_2 v_2 + 2u_3 v_3$. 

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Let $a = (1,2,1)$ and $b = (-1,1,1)$.

a. Determine the angle of $a$ with $b$, for each inner product
b. Determine the norm of $a$ and $b$, for each inner product

3. In the real vector space $\mathbb{R}^3$, there is the following inner product:

$$\forall u = (u_1,u_2,u_3), v = (v_1,v_2,v_3),$$

$$u \cdot v = 2u_1v_1 + u_2v_1 - u_3v_1 + u_1v_2 + 3u_2v_2 - u_1v_3 + u_3v_3.$$

a. Show that the actual $\alpha$ values are orthogonal vectors $a = (2,\alpha,1)$ and $b = (\alpha+1,2,-1)$.

b. Determine a unitary and orthogonal vector $u = (1,-1,2)$ and $v = (2,1,-1)$.

4. In the real vector space $\mathbb{R}^3$, let

$$B = \{w_1 = (1,-1,1), w_2 = (1,1,0), w_3 = (-1,1,2)\}.$$

a. Show that $B$ is formed by orthogonal vectors, relative to the standard inner product.

b. Conclude if $B$ is or not base of $\mathbb{R}^3$.

c. Determine a set of orthonormal vectors $C = \{u_1, u_2, u_3\}$ formed by scalar vector of $B$.

d. Write the vector $v = (x,y,z)$, $x,y,z \in \mathbb{R}$, as a linear combination of the vectors.

5. In the vector space $\mathbb{P}^3$, consider the inner product defined by:

$$\forall u = p(x), v = q(x) \in \mathbb{P}^3, u \cdot v = \int_{-1}^{1} p(t)q(t)dt.$$

a. Verify that $B = \{x^3, x^2, x, 1\}$ is an orthogonal of base $\mathbb{P}^3$.

b. Calculate the projection of $p(x) = x^3$ in $q(x) = x$.

c. Show that $p(x) = x^2$ and $q(x) = x$ are orthogonal and check the Pythagorean Theorem.

d. Determine the angle $\theta$ between $p(x) = x^3$ and $q(x) = x$.

Inner Product in Finite-Dimensional Space

Definition (Euclidean Space). A real vector $V$, with an inner product and finite-dimensional product, denoted by the Euclidian Space.

Definition (Orthogonal and Orthonormal Basis). Let $V$ be a euclidian space of dimension $n \in \mathbb{N}$.

We say that:

a. A base $B$ of $V$ is orthogonal if $B$ is an orthogonal system of vectors.

b. A base $B$ of $V$ is orthonormal if $B$ is an orthonormal system of vectors.
Exemple. In the real vector space $\mathbb{R}^n$, the standard basis, that is, the base formed by the vector:

$$e_1 = (1,0,0,...,0), e_2 = (0,1,0,...,0), ..., e_n = (0,0,0,...,1),$$

is compared to the usual orthonormal inner product $(or$ $standard)$, i.e.,

$$\forall u = (u_1, u_2, u_3, ..., u_n), v = (v_1, v_2, v_3, ..., v_n) \in \mathbb{R}^n, \text{ it has been}$$

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 + ... + u_n v_n.$$ 

Theorem. Let $w_1, w_2, ..., w_n$ be a non-zero orthogonal vector of $V$. Then, $\forall v \in V$, the vector

$$v - \text{proj}_{w_1} v - ... - \text{proj}_{w_n} v$$

is orthogonal to $w_k$, for each $k \in \{1, 2, ..., n\}$.

Theorem. Let $B = \{e_1, e_2, ..., e_n\}$ a basis of a Euclidian space $V$, then, there exist a basis $B' = \{u_1, u_2, ..., u_n\}$ of $V$ is orthonormal and

$$u_1 = e_1 / \|e_1\|.$$ 

Note. In the construction of the base $B' = \{u_1, u_2, ..., u_n\}$, from the previous theorem, first the process of Gram-Schmidt of orthogonalisation is used to form an orthogonal basis, as follows:

$$w_1 = e_1$$

$$w_2 = e_2 - \text{proj}_{w_1} e_2$$

$$w_3 = e_3 - \text{proj}_{w_1} e_3 - \text{proj}_{w_2} e_3$$

$$w_n = e_n - \text{proj}_{w_1} e_n - \text{proj}_{w_2} e_n - ... - \text{proj}_{w_{n-1}} e_n$$

Then, the inverse is determined for vector $w_i \in \{1, 2, ..., n\}$ obtained:

$$u_1 = w_1 / \|w_1\|, u_2 = w_2 / \|w_2\|, u_3 = w_3 / \|w_3\|, ..., u_n = w_n / \|w_n\|.$$ 

Exemple. Be a real vector space $\mathbb{R}^3$, with standard inner product:

$$\forall u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3, u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3.$$ 

Consider the basis $B = \{(1,1,1),(0,0,1),(1,0,0)\}$ of $\mathbb{R}^3$. Determine an orthonormal basis $B' = \{u_1, u_2, u_3\}$, where

$$u_1 = (1,1,1) / \|(1,1,1)\|.$$ 

then, it will first be orthogonalised the basis $B$, applying the orthogonalisation process Gram-Schmitd:

$$w_1 = (1,1,1)$$

$$w_2 = (0,0,1)-(0,0,1)-(1,1,1)/\|(1,1,1)\| \cdot 2 (1,1,1) = (0,0,1)-(1/3,1/3,1/3) = (-1/3,-1/3,2/3).$$

$$w_3 = (1,0,0)-(1,0,0)-(1,1,1)/\|(1,1,1)\| \cdot 2 (1,1,1)-((1,0,0)-(1/3,-1,3,2/3)/\|(1,1,1)\|$$
Then:

\[ \begin{align*}
\mathbf{u}_1 &= \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \sqrt{3}/3(1,1,1) = (\sqrt{3}/3,\sqrt{3}/3,\sqrt{3}/3) \\
\mathbf{u}_2 &= \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \sqrt{6}/2((-1)/3,(-1)/3,2/3) = (-\sqrt{6}/6,-\sqrt{6}/6,2\sqrt{6}/6) \\
\mathbf{u}_3 &= \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \sqrt{2}(1/2,(-1)/2,0) = (\sqrt{2}/2,(-\sqrt{2}/2,0) \\
\end{align*} \]

Therefore, \( B^* = \{(\sqrt{3}/3,\sqrt{3}/3,\sqrt{3}/3),(-\sqrt{6}/6,-\sqrt{6}/6,2\sqrt{6}/6),\sqrt{2}/2,(-\sqrt{2}/2,0)\} \).

Definition (Metric Matrix). Let \( V \) be Euclidean space \( n \in \mathbb{N} \) dimension, and let \( B = \{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\} \) be its basis. The real matrix \( G = [\mathbf{e}_i \cdot \mathbf{e}_j], i,j \in \{1,2,\ldots,n\} \), is called a metric matrix (the inner product defined in \( V \)) from the base considered.

Note. In Euclidean space \( V \), a metric matrix relative to the base orthonormal \( B = \{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\} \) is the identity matrix, then:

\( \mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \),

because when \( i \neq j \), the vectors \( \mathbf{e}_i \) and \( \mathbf{e}_j \) are proportional and when \( i = j \), \( \|\mathbf{e}_i\| = 1 \iff \mathbf{e}_i \cdot \mathbf{e}_i = 1 \).

Theorem. Let \( V \) be an Euclidean space \( n \in \mathbb{N} \) dimension, and let \( B = \{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\} \) be its basis. Let \( G = [\mathbf{e}_i \cdot \mathbf{e}_j], i,j \in \{1,2,\ldots,n\} \) be the matrix of metric (the inner product defined on \( V \)) relative to the base \( B = \{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\} \). Let \( \mathbf{u}, \mathbf{v} \in V \) such that

\[ \mathbf{u} \equiv_B (u_1,u_2,\ldots,u_n) \quad \text{and} \quad \mathbf{v} \equiv_B (v_1,v_2,\ldots,v_n) \] .

Then,

\[ \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} G \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [\mathbf{u} \cdot \mathbf{v}] . \]

Example. \( \forall \mathbf{u} = (u_1,u_2,u_3), \mathbf{v} = (v_1,v_2,v_3) \in \mathbb{R}^3 \), define the following inner product in real vector space \( \mathbb{R}^3 \):

\[ \mathbf{u} \cdot \mathbf{v} = 2u_1 v_1 + u_2 v_2 + u_3 v_3 + u_2 v_1 + 3u_2 v_2 + u_3 v_1 + u_1 v_2 + 3u_1 v_2 + u_2 v_3 + u_3 v_3 . \]

1. Regarding the standard basis \( \mathbb{R}^3 \), the inner product can be represented in matrix form by:

\[ \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [\mathbf{u} \cdot \mathbf{v}] . \]

Note. As \( \forall \mathbf{u} = (u_1,u_2,u_3) \equiv_B (u_1,u_2,u_3) \), then you can define an inner product using the metric matrix with respect to the standard basis.
Theorem. Let $V$ be a Euclidean space, let $B=\{v_1, v_2, \cdots, v_n\}$ and $B^\prime=\{v_1^\prime, v_2^\prime, \cdots, v_n^\prime\}$ two bases of $V$, and let $P=M(B \rightarrow B^\prime)$, matrix base change, the old base $B$ to the new base $B^\prime$. If $G$ is the metric matrix of the inner product $V$, in relation to the base $B$, then the metric matrix inner product $V$, in relation to the base $B^\prime$, is
\[ G^\prime = P^\top G P . \]

Consequence of The Previous Theorem. If $B=\{v_1, v_2, \cdots, v_n\}$ and $B^\prime=\{v_1^\prime, v_2^\prime, \cdots, v_n^\prime\}$ are orthonormal bases of $V$, then $G^\prime = G = I_n$, therefore we have:
\[ [I_n] = P^\top P \Rightarrow P^{-1} = P^\top. \]

Definition. (Orthogonal Matrix). Let $P$ be a real square matrix of order $n$. It is said that $P$ is orthogonal, if $P^{-1} = P^\top$.

Example. Consider the real vector space $\mathbb{R}^3$, with standard inner product, i.e., for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$,
\[ x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3. \]
Let $B=\{(1,0,0),(0,1,0),(0,0,1)\}$ and $B^\prime=\{(1/3,2/3,-2/3),(2/3,1/3,2/3),(-2/3,2/3,1/3)\}$ orthonormal bases of $\mathbb{R}^3$. Let
\[ P = M(B \rightarrow B^\prime) = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}. \]
By the previous theorem, $P$ is an orthonormal matrix. In fact, $P^\top P = I_3$, which is to say that $P^{-1} = P^\top$.

Theorem (Product of Orthogonal Matrix). Let $P$ and $Q$ be real square matrices. If $P$ and $Q$ are orthogonal matrices, then $PQ$ is an orthogonal matrix.

Definition (Positive Definite Matrix). Let $A$ be a real symmetric matrix, i.e., $A = A^\top$. It is said that $A$ is a positive definite matrix, for each vector $0 \neq u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, we have:
\[ u \cdot u = \sum_{i=1}^{n} u_i^2 > 0. \]

Note. As $\forall u = (u_1, u_2, u_3) \equiv_B (u_1, u_2, u_3)$, then you can define an inner product using the metric matrix with respect to the standard basis.

Theorem. Let $V$ be a Euclidean space, let $B=\{v_1, v_2, \cdots, v_n\}$ and $B^\prime=\{v_1^\prime, v_2^\prime, \cdots, v_n^\prime\}$ two bases of $V$, and let $P=M(B \rightarrow B^\prime)$, matrix base change, the old base $B$ to the new base $B^\prime$. If $G$ is the metric matrix of the inner product $V$, in relation to the base $B$, then the metric matrix inner product $V$, in relation to the base $B^\prime$, is
\[ G^\prime = P^\top G P . \]

Consequence of The Previous Theorem. If $B=\{v_1, v_2, \cdots, v_n\}$ and $B^\prime=\{v_1^\prime, v_2^\prime, \cdots, v_n^\prime\}$ are orthonormal bases of $V$, then $G^\prime = G = I_n$, therefore we have:
\[ [I_n] = P^\top P \Rightarrow [P^\top]^\prime = P^\top . \]
Definition. (Orthogonal Matrix). Let $P$ be a real square matrix of order $n$. It is said that $P$ is orthogonal, if $P^{-1}=P^t$.

Example. Consider the real vector space $\mathbb{R}^3$, with standard inner product, i.e., for all $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3) \in \mathbb{R}^3$, $x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$.

Let $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ and $B' = \{(1/3,2/3,-2/3),(2/3,1/3,2/3),(-2/3,2/3,1/3)\}$ orthonormal bases of $\mathbb{R}^3$. Let

$$P = M(B \rightarrow B') = \begin{bmatrix}
    1/3 & 2/3 & -2/3 \\
    2/3 & 1/3 & 2/3 \\
    -2/3 & 2/3 & 1/3 
\end{bmatrix}.$$

By the previous theorem, $P$ is an orthonormal matrix. In fact, $PP^t = I_3$, which is to say that $P^{-1} = P^t$.

Theorem (Product of Orthogonal Matrix). Let $P$ and $Q$ be real square matrices. If $P$ and $Q$ are orthogonal matrices, then $PQ$ is an orthogonal matrix.

Definition (Positive Definite Matrix). Let $A$ be a real symmetric matrix, i.e., $A=A^t$. It is said that $A$ is a positive definite matrix, for each vector $0 \neq u = (u_1,u_2,\ldots,u_n) \in \mathbb{R}^n$, we have:

$$\text{Theorem. Let a real symmetric matrix of order } 2 \text{ be } A = [a \ b \ b \ c].$$

Then, the matrix $A$ is positive definite if and only if, $|A|=ac-b^2>0$.

Example. A real matrix $A = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & 5 & 5 \end{bmatrix}$ is positive definite.

Theorem. Let $A$ be a positive definite real matrix. Then $A$ define an inner product on $\mathbb{R}^n$, that is, for all vector $u = (u_1,u_2,\ldots,u_n)$, $v = (v_1,v_2,\ldots,v_n)$ of $\mathbb{R}^n$:

$$\text{Theorem. Let } V \text{ be an Euclidian space. Then any matrix of the metric is positive definite.}$$

Definition (Orthogonal Complement). Let $V$ be a Euclidean space, and is $\emptyset \neq S \subseteq V$. The orthogonal complement of $S$, designated by $S^\perp$, is the set of all vectors $V$ that are orthogonal to all vectors of $S$, or,

$$S^\perp = \{v \in V : v \cdot u = 0, \forall u \in S\}.$$
Theorem. Let $V$ be a Euclidean space, and is $\emptyset \neq S \subseteq V$. Then, $S^\perp$ is a vector subspace $V$.

Exemple. Let $V$ be a Euclidean space. If $S=\{0 \vec{v}\}$, then $S^\perp=V$. And if $S=V$, then $S^\perp=\{0 \vec{v}\}$.

Exemple. In Euclidean space $R^n$, $n \in \mathbb{N}$ and $n>1$. Let $S=\{0 \vec{v}\}$, where $0 \vec{v}=(a_1,a_2,\ldots,a_n) \in R^n$. Then,

$$S^\perp=\{x \vec{v}=(x_1,x_2,\ldots,x_n) \in R^n : \vec{v} \cdot x \vec{v}=0\}
\iff\{(x_1,x_2,\ldots,x_n) \in R^n : a_1 x_1+a_2 x_2+\cdots+a_n x_n=0\}.$$

Special Case of the Euclidean Space $R^3$. If $S=\{0 \vec{v}\}$, where $0 \vec{v}=(a,b,c) \in R^3$, then,

$$S^\perp=\{(x,y,z) \in R^3 : ax+by+cz=0\},$$

the plane equation which passes through the origin and the normal vector $\vec{v}=(a,b,c)$.

Special Case of the Euclidean Space $R^2$. If $S=\{0 \vec{v}\}$, where $0 \vec{v}=(a,b) \in R^2$, then,

$$S^\perp=\{(x,y) \in R^2 : ax+by=0\},$$

the equation of the line through the origin.

Note (Very important!!!). In Euclidean space $R^n$, with standard inner product, let $S=\{0 \vec{v}_{-1},0 \vec{v}_{-2},\ldots,0 \vec{v}_{-m}\}$, where

$$0 \vec{v}_{-i}=(a_{i1},a_{i2},\ldots,a_{in}), i=1,2,\ldots,n.$$

Then, $S^\perp$ is the vector subspace of $R^n$, defined pelby the homogeneous system

$$\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
=\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.$$

Exemple. In Euclidean space $R^3$, with standard inner product, let $S=\{(1,1,1),(-1,0,1)\}$.

Determine $S^\perp$.

Then,

$$S^\perp=\{(x,y,z) \in R^3 : x+y+z=-x+z=0\}$$

$$S^\perp=\{(\lambda,-2\lambda,\lambda) : \lambda \in \mathbb{R}\}.$$

$$S^\perp=\{(\lambda,-2\lambda,\lambda) : \lambda \in \mathbb{R}\}=(1,-2,1)).$$
Theorem. Let V be a Euclidean space with an inner product. For any non-empty subsets X and Y to V, we have:

a. If X⊆Y, then Y^⊥⊆X^⊥.

b. X⊆[(X^⊥)]^⊥.

c. [(X)]^⊥=X^⊥; and in particular, [(⟨v^⊥_1,v^⊥_2,...,v^⊥_k⟩)]^⊥=⟨⟨v^⊥_1,v^⊥_2,...,v^⊥_k⟩⟩^⊥.

Note. The “point 3.” of the last theorem is used to calculate orthogonal component of any subspace of V finitely generated.

Exemple. In Euclidean space R^3, let S=⟨(1,1,1),(-1,0,1)⟩. Then,

S^⊥=⟨⟨(1,1,1),(-1,0,1)⟩⟩^⊥=⟨(1,1,1),(-1,0,1)⟩^⊥=⟨(1,-2,1)⟩.

Theorem. Let V be a Euclidean space, and let F be a vector subspace of V. There is, in this case:

1. V=F⨁F^⊥.

2. [(F^⊥)^⊥]=F.

Definition. Let V be a Euclidian space, and let F be a vector subspace of V. Given that v"∈V, it can be written v"=v"_1+v"_2, where v"_1∈F and v"_2∈F^⊥. Under these conditions, v"_1 is the orthogonal projection on F and v"_2 is the orthogonal projection on F^⊥, and writes:

v"_1=proj^⊥_F v"  and  v"_2=proj^⊥_(F^⊥) v".

Exemple. In Euclidean space R^3, with the canonical inner product, let the vector subspace F=⟨(1,1,0)⟩. Let v"=(x,y,z), be an arbitrary vector R^3. Determine the orthogonal projection of v" on F, and on F^⊥.

Now,

F=[α(1,1,0)∶ α∈R]=⟨(α,-α,0)∶ α∈R⟩.

Determine first of all the orthogonal complement of F. However,

F^⊥ = ⟨{a,b,c}∈R^3∶ (α,-α,0)∙(a,b,c)=0⟩

= ⟨{a,b,c}∈R^3∶ αa-αb=0⟩

= ⟨{a,b,c}∈R^3∶ a-b=0⟩

= ⟨{b,b,c}∶ b,c∈R⟩

= ⟨⟨1,1,0⟩,(0,0,1)⟩

As R^3=F⨁F^⊥, then R^3=⟨⟨1,1,0⟩,(0,0,1)⟩, which means that

B=[⟨1,1,0⟩,(0,0,1)⟩

Is a basis of R^3.

Therefore, there are real scalar α,β,δ, such that

v"=(x,y,z)=α(1,1,0)+β(1,1,0)+δ(0,0,1).
then:
\[ \alpha = \frac{1}{2}(x-y), \beta = \frac{1}{2}(x+y), \delta = z. \]

Hence,
\[ \text{pro}_{j^\perp} v = \left( \frac{x-y}{2}, \frac{y-x}{2}, 0 \right) \quad \text{and} \quad \text{pro}_{j^\perp} (F^\perp) v = \left( \frac{x+y}{2}, \frac{x+y}{2}, z \right). \]

**Theorem.** Let \( B = \{v_1^\perp, v_2^\perp, \ldots, v_k^\perp\} \) be an orthogonal basis of the vector subspace \( F \), of a vector space with an inner product. Then
\[ \text{pro}_{j^\perp} \left( F^\perp \right) v = \text{pro}_{j^\perp} (v_1^\perp) v + \text{pro}_{j^\perp} (v_2^\perp) v + \cdots + \text{pro}_{j^\perp} (v_k^\perp) v. \]

**Definition (Distance between two vectors).** Let \( V \) be a vector space with an inner product. For any vectors \( a, b \in V \), there is a real number called distance from \( a \) to \( b \) such that:
\[ d(a, b) = \|a - b\|. \]

**Theorem (Properties of distance between two vectors).** Let \( V \) be a real vector space with inner product. For any vectors \( a, b, c \in V \), we have:
1. \( d(a, b) = 0 \iff a = b \).
2. \( d(a, b) = d(b, a) \).
3. \( d(a, b) \leq d(a, c) + d(c, a) \).

**Theorem.** Let \( V \) be a vector space with inner product and let \( F \) be a vector subspace of finite dimension of \( V \). For each vector \( v \in V \), the \( \text{pro}_{j^\perp} v \) is a vector of \( F \) closer to \( v \), that is, its distance to \( v \) is minimal.

**Definition (Distance from a vector to a subspace).** Let \( V \) be a vector space with inner product, and let \( F \) be a vector subspace of finite dimension of \( V \). Given a vector \( v \in V \), there is a real number called distance from \( v \) to \( F \), such that
\[ d(v, F) = \min \{ \|v - f\| : f \in F \}. \]

**Note.** Let \( V \) be a vector space with inner product and let \( F \) be a vector subspace of finite dimension of \( V \). By the previous theorem, for each \( v \in V \), then:
\[ d(v, F) = \min \{ \|v - f\| : f \in F \} = \|v - \text{pro}_{j^\perp} v\| = \sqrt{\|v\|^2 - \|\text{pro}_{j^\perp} v\|^2} \].

The last equality is due to the Pythagorean Theorem.

**Outer Product**

**Theorem.** Let \( V \) be a Euclidean space of three dimension. Let the vectors \( u, v \in V \) be linearly independent. Then there exist an infinite vectors \( z \), such that:
\[ z \perp u \quad \text{and} \quad z \perp v. \]

**Theorem.** Let \( V \) be a Euclidean space of three dimension. Let the vectors \( u, v \in V \) be linearly independent. Let \( \epsilon \) be any real positive number. Then there are two and only two vectors \( z \), such that:

a. \( z \perp u \quad \text{and} \quad z \perp v \),

b. \( z \perp u \),

...
b. \( \|z\|=\varepsilon \).

Definition. Let \( V \) be a real vector space of dimension \( n \in \mathbb{N} \). Let \( B=\{e_1,e_2,\ldots,e_n\} \) be a basis of \( V \). Let \( v_1,v_2,\ldots,v_n \), be arbitrary vectors of \( V \). Let the matrix

\[
A = \begin{bmatrix}
v_{11} & v_{12} & \cdots & v_{1n} \\
v_{21} & v_{22} & \cdots & v_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
v_{n1} & v_{n2} & \cdots & v_{nm}
\end{bmatrix}
\]

whose columns are composed of the components of the vectors \( v_1,v_2,\ldots,v_n \), the basis \( B \) considered. The determinant of the matrix \( A \), takes the name of the determinant of vectors \( v_1,v_2,\ldots,v_n \) with respect to the basis \( B \):

\[
\text{det}_B (v_1,v_2,\ldots,v_n)=|A|.
\]

Teorema. Let \( V \) be a Euclidean space of dimension 3, with fixed basis \( B=\{e_1,e_2,e_3\} \). Let \( \varepsilon \) be any positive real number and let the vectors \( u,v \in V \) linearly independent. Then, there exist one and only one vector \( z \) such that:

a. \( z \perp u \) and \( z \perp v \),

b. \( \|z\|=\varepsilon \),

c. \( \text{det}_B (u,v,z)>0 \)

Definition (Outer Product). Let \( V \) be an Euclidean space of dimension 3 with a fixed basis \( B=\{e_1,e_2,e_3\} \). For any vectors \( u,v \in V \), is called outer product (or vectorial product) of \( u \) by \( v \), and is represented by \( u \times v \), the vector is defined as:

1. If the vectors \( u \) and \( v \) are linearly dependent, then \( u \times v = 0 \).
2. If the vectors \( u \) and \( v \) are linearly independent, then \( u \times v \) is the only vector \( V \) that verifies the following conditions

a. \( u \times v \perp u \) and \( u \times v \perp v \),

b. \( \|u \times v \|=\|u\|\|v\| \sin(\angle(u,v)) \),

c. \( \text{det}_B (u,v,u \times v)>0 \).

Note. The determination of outer product from two linearly independent vectors is in general very laborious. However, if one knows the components of the vectors concern with respect to an orthonormal basis, the calculation is much easier according to the following theorem.

Theorem. Let \( V \) be an Euclidean space of dimension 3, with a fixed orthonormal basis sets \( B=\{e_1,e_2,e_3\} \). For any vectors \( u,v \in V \), there are

\[
u \equiv_B (u_1,u_2,u_3) \quad \text{e} \quad v \equiv_B (v_1,v_2,v_3) .
\]

So, we have:

\[
u \times v=(u_2 v_3-u_3 v_2,u_3 v_1-u_1 v_3,u_1 v_2-u_2 v_1) .
\]
Note. There exist mnemonic rule for the result of the previous theorem:
\[ \mathbf{u} \times \mathbf{v} = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = e_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - e_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + e_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}, \]

Applying Laplace development on the first line.

Note. If the vectors \( \mathbf{u} \) and \( \mathbf{v} \) are vectors of the Euclidian space \( \mathbb{R}^3 \), then \( ||\mathbf{u} \times \mathbf{v}|| \) represent the area of the parallelogram, constructed on the data vectors, like the figure below.

Exemple. In the Euclidian space of \( \mathbb{R}^3 \), with canonical inner product, there are the vectors \( \mathbf{u} = (1, 3, 4) \) and \( \mathbf{v} = (2, -6, -5) \) of \( \mathbb{R}^3 \).

a. Determine a unitary and orthogonal vector \( \mathbf{u} \) and \( \mathbf{v} \).

b. Determine the area of the parallelogram constructed on the vectors \( \mathbf{u} \) and \( \mathbf{v} \).

Paragraph (a). Let \( B = \{(1,0,0),(0,1,0),(0,0,1)\} \) be an orthonormal basis of \( \mathbb{R}^3 \), with respect to the canonical inner product. Then,
\[ \mathbf{u} =_B (1,3,4) \quad \text{and} \quad \mathbf{v} =_B (2,-6,-5). \]
\[ \mathbf{u} \times \mathbf{v} = (|3 4 -6 -5 |,-|1 4 2 -5 |,|1 3 2 -6 |) = (9,13,-12). \]
The vector requested is the inverse of \( \mathbf{u} \times \mathbf{v} \):
\[ 1/(||\mathbf{u} \times \mathbf{v}||) \mathbf{u} \times \mathbf{v} = ((9/394),(13/394),(12/394)/394). \]

Paragraph (b). The area of the parallelogram is \( 394 \mathbf{u}^2 \).

Conclusion

The introduction of inner product in real abstract vector space allows, in a certain way, add further characteristic of the vectors (normal, angle between vectors, orthogonal projection, etc.)

The matrix base change, among other things, allows to relate different aspects among different bases in a real finite dimensional vector space.

The metric matrix lets you know if a particular base or not orthogonal (or orthonormal) for internal product concerned.
The external product has many applications in the area and volume calculation in Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$

**Assessment**

1. In the real vector space $\mathbb{P}^2$, let the basis $B=\{x^2-x+1,-x+1,-x^2-x+2\}$ and $B'=\{x^2-x,-x^2+2x+1,-x^2+x+1\}$.
   a. Determine $P=M(B\rightarrow B')$.
   b. Show that $Q=P^{-1}=M(B'\rightarrow B)$.

2. Determine the components $(\lambda_1,\lambda_2,\lambda_3)$ of $v^- = ax^2+bx+1$, $a,b \in \mathbb{R}$, in the basis $B$.

3. Show that the components $([\lambda'_1],[\lambda'_2],[\lambda'_3])$ of vector $v^- \text{ in the basis } B'$, is given by:
   $[\lambda'_1 \lambda'_2 \lambda'_3 ] = Q[\lambda_1 \lambda_2 \lambda_3 ]$.
   a) Show that the following operation is an inner product of the vector space given
   b) $\forall u^-=(u_1,u_2), v^-=(v_1,v_2) \in \mathbb{R}^2, u^-\cdot v^- = 3u_1 v_1+u_2 v_1+u_1 v_2+2u_2 v_2$.

3. Consider the following inner product in the real vector space $\mathbb{R}^3$:
   $\forall u^-=(u_1,u_2,u_3), v^-=(v_1,v_2,v_3), u^-\cdot v^- = 2u_1 v_1+u_3 v_1+2u_2 v_2-u_3 v_2+u_1 v_3-u_2 v_3+2u_3 v_3$.
   Let $a^-=(1,2,1)$ and $b^-=(-1,1,1)$.
   (a) Determine the angle of $a^-$ with $b^-$ with respect to the inner product
   (b) Determine a normal of $a^-$ and $b^-$, with respect to the inner product

4. In the real vector space $\mathbb{R}^3$, let the following inner product:
   $\forall u^-=(u_1,u_2,u_3), v^-=(v_1,v_2,v_3), u^-\cdot v^- = 2u_1 v_1+u_3 v_1+u_1 v_2+3u_2 v_2-u_1 v_3+u_3 v_3$.
   (a) Show that the real values of $\alpha$ are orthogonal to the vectors $a=(2,\alpha,1)$ and $b$
   (b) Determine a unit and orthogonal vector $u^-=(1,-1,2)$ and $v^-=(2,1,-1)$

5. In the real vector space $\mathbb{R}^3$, let
   $B=\{w^-_1=(1,-1,1), w^-_2=(1,1,0), w^-_3=(-1,1,2)\}$.
   (a) Show that $B$ is formed by orthogonal vector with respect to the canonical inner product.
   (b) Conclude if $B$ is or is not in the basis of $\mathbb{R}^3$.
   (c) Determine a orthonormal set of vectors $C=\{u^-_1, u^-_2, u^-_3\}$ formed by the scalar multiple of the vectors of $B$.
   (d) Write the vector $v^-=(x,y,z), x,y,z \in \mathbb{R}$, with linear combination of vectors of $B$, without solving the system
6. Consider the Euclidean space $\mathbb{R}^4$, with the euclidean inner product. Use the Gram-Schmidt process of orthogonalization to find an orthonormal basis of vector subspace $F$ with basis $B = \{(1,1,-1,0),(0,2,0,1),(-1,0,0,1)\}$.

7. Let the Euclidean space $\mathbb{R}^3$, with the following inner product:

$$\langle \vec{u}, \vec{v} \rangle = (u_1, u_2, u_3, v_1, v_2, v_3) \in \mathbb{R}^3,$$

$$\vec{u} \cdot \vec{v} = 2u_1 v_1 + u_2 v_1 - u_3 v_1 + u_1 v_2 + 3u_2 v_2 - u_1 v_3 + u_3 v_3.$$

(a) Determine the metric matrix $G$ with respect to the canonical basis $B$, and the metric matrix $G'$, with respect to the basis $B' = \{(1,1,1),(-1,0,1),(2,1,1)\}$, of $\mathbb{R}^3$.

(b) Determine the matrix basis change $P=M(B \rightarrow B')$.

(c) Show that $G' = P^t GP$.

(d) Show that $G'$ is a positive define matrix.

8. In the euclidean space $\mathbb{R}^2$, let the following inner product:

$$\vec{x} \cdot \vec{y} = 3x_1 y_1 - x_1 y_2 - x_2 y_1 + 2x_2 y_2,$$

where $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ are arbitrary vectors of $\mathbb{R}^2$. Find the orthonormal basis for the space.

9. In the euclidean space $\mathbb{R}^3$, let an inner product with respect to the canonical basis, has the following metric matrix

$$G = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$ 

Find an orthonormal basis for $\mathbb{R}^3$.

10. In the euclidean space $\mathbb{R}^3$, with the canonical inner product, let the vectors

$$\vec{u} = (-1, 1, 2), \quad \vec{v} = (0, -1, 2) \quad \vec{e} = (2, -1, -3).$$

Determine, assuming fixed canonical basis:

a. $\vec{u} \times \vec{v}$

b. $\vec{v} \times \vec{w}$

c. $(\vec{u} + \vec{v}) \times \vec{w}$

d. $\vec{u} \cdot (\vec{v} \times \vec{w})$

Let $\vec{x} = \vec{e}_1 - \vec{e}_3$, $\vec{y} = -\vec{e}_1 + \vec{e}_2$, $\vec{z} = -\vec{e}_1 + 2\vec{e}_3$. Calculate:

a. $\vec{x} \times \vec{y}$

b. $(\vec{x} \times \vec{y}) \cdot \vec{z}$

c. $(\vec{x} \times \vec{y}) \times \vec{z}$. 


Activity 2.3 - Real vector space: Definition and properties; Basis and dimension

Introduction

This activity addresses the concept of vector space and its main properties. It gives prominence to the real vector space finitely generated, with emphasis on the space $\mathbb{R}^n$, $n \in \mathbb{N}$ and $n>1$, and $\mathbb{P}^n$, $n \in \mathbb{N}$.

It approaches the dependency concept and linear independence because they are associated with all properties of a vector space.

We will give special attention to the basic study and dimension of a vector space.

The introduction of the inner product on a vector space, calculating the norm of vector, calculating the angle between two vector, determining the orthogonal projection of a vector onto another vector and a vector subspace.

Activity Details

Definition (Real Vector Space). Let $V$ be a non-empty set and let $F$ be a field. It is said that $V$ is a vector space on the field $F$ when:

$V$ is defined as a set of operation called addition and represented by “+”, such that,

$\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} \in V$,

and which goes with the following properties:

1. $\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} = \vec{v} + \vec{u}$;
2. $\forall \vec{u}, \vec{v}, \vec{w} \in V, \vec{u} + (\vec{w} + \vec{v}) = (\vec{u} + \vec{w}) + \vec{v}$;

There exist a vector in $V$, denoted “zero vector”, and is denoted by $0$, such that

3. $\forall \vec{u} \in V, \vec{u} + 0 = \vec{u}$;
4. $\forall \vec{u} \in V$, there exist a vector in $V$, which is called “symmetric of $\vec{u}$”, and is denoted by “$-\vec{u}$”, such that $\vec{u} + (-\vec{u}) = 0$.

Product (or multiplication), “denoted by “product (or multiplication) by scalar”, such that $\forall \vec{u} \in V, \forall \lambda \in F, \lambda \vec{u} \in V$,

and goes with the following properties:

1. $\forall \lambda \in F, \forall \vec{u}, \vec{v} \in V, \text{we have } \lambda(\vec{u} + \vec{v}) = \lambda \vec{u} + \lambda \vec{v}$;
2. $\forall \lambda, \beta \in F, \forall \vec{u} \in V, \text{we have } (\lambda + \beta) \vec{u} = \lambda \vec{u} + \beta \vec{u}$;
3. $\forall \lambda, \beta \in F, \forall \vec{u} \in V, \text{we have } (\lambda \beta) \vec{u} = \lambda(\beta \vec{u})$;
4. $\forall \vec{u} \in V, \text{tem-se } 1 \vec{u} = \vec{u}$.
When these conditions are satisfied, the elements of V are called vectors, and that of F, for scalars.

In the field F=R, the vector space is said to be real.

Exemplo. Let \( n \in \mathbb{N} \) and \( n > 1 \). Consider the set

\[ R^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, n \} \]

with the following operation:

Addition (usual): \( \forall \ x \vec{=} = (x_1, x_2, \ldots, x_n), y \vec{=} = (y_1, y_2, \ldots, y_n) \in R^n, x \vec{+} y \vec{=} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \),

Multiplication by scalar (usual): \( \forall \ x \vec{=} = (x_1, x_2, \ldots, x_n) \in R^n, \forall \lambda \in \mathbb{R}, \lambda x \vec{=} = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n) \).

The operation of \( R^n \) is a real vector space. This activity study only real vector space.

Note that the zero vector is \( 0 \vec{=} = (0, 0, \ldots, 0) \), and symmetric of a vector \( x \vec{=} = (x_1, x_2, \ldots, x_n) \in R^n \) is \( -x \vec{=} = (-x_1, -x_2, \ldots, -x_n) \in R^n \).

Example. Let \( P \) be the set of all polynomials with real coefficients and one unknown. \( P \), with the typical addition of polynomial, and the typical multiplication of a real number by a polynomial, is a real vector space.

Example. Let \( n \in \mathbb{N}_0 \) (set of all non-negative numbers). Let \( P^n \) be the set of all polynomials with real coefficients and one unknown variable of degree not greater than \( n \), that is,

\[ P^n = \{a_n x^n + a_{(n-1)} x^{(n-1)} + \cdots + a_1 x + a_0 : a_i \in \mathbb{R}, i = 0, n\} \],

\( P^n \), with the typical addition of these polynomial, and the typical multiplication of a real number by polynomial, is a real vector space.

Note. The set \( P^0 = \mathbb{R} \), that is, the set \( R \) is a real vector space, with an addition of real numbers and the typical multiplication of a real number by a real number.

The zero vector in vector space \( P \) and \( P^n \), \( n \in \mathbb{N}_0 \), is \( 0 \vec{=} = 0 \), and the symmetric of a vector

\[ v \vec{=} = p(x) = a_n x^n + a_{(n-1)} x^{(n-1)} + \cdots + a_1 x + a_0 \]

is

\[ -v \vec{=} = -p(x) = -a_n x^n - a_{(n-1)} x^{(n-1)} - \cdots - a_1 x - a_0 \].

Where references to the vector spaces \( R^n \), \( P \) and \( P^n \), not to mention the operations involved, should be considered the normal operations in these vector spaces.

Definition (Subtraction). A subtraction of vectors \( u \vec{=} \) and \( v \vec{=} \) of a real vector \( V \), \( u \vec{=} -v \vec{=} \), is defined by:

\[ u \vec{=} -v \vec{=} = u \vec{=} + (-v \vec{=} ) \].
Theorem (Properties of a real vector space). Let $V$ be a real vector space. For any vectors $u, v \in V$, and any real scalar $\lambda \in \mathbb{R}$, we have:

1. $0v = 0$;
2. $\lambda 0 = 0$;
3. $\lambda v = 0$ if and only if, $\lambda = 0$ or $v = 0$;
4. $(\lambda \lambda')v = (\lambda' \lambda)v$;
5. $\lambda(u - v) = \lambda u - \lambda v$;
6. $(\lambda - \lambda')u = \lambda u - \lambda' u$.

Definition (Linear Combination). Let $V$ be a real vector space. It is that a vector $v \in V$ is a linear combination of vectors $v_1, v_2, \ldots, v_n \in V$, if there exist a real scalar $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$.

Example. In the real vector space $\mathbb{R}^3$ (with the typical operation), the vector $v = (-2, 2, 5)$ is a linear combination of the vectors $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$ and $v_3 = (1, 0, 1)$?

However, $v$ is a linear combination of $v_1$, $v_2$ and $v_3$ if and only if the equation

$v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$, with unknown $\lambda_i$, $i = 1, 3$, is possible (fixed or indefinite).

$v = [\lambda_1 v_1] + \lambda_2 [v_2] + [\lambda_3 v_3] \iff [1 1 1 1 1 0 1 0 0] [\lambda_1 \lambda_2 \lambda_3] = [-2 2 5] \iff \{\lambda_1 = 9, \lambda_2 = -7, \lambda_3 = 4\}$.

then, $v$ is a linear combination of $v_1$, $v_2$ and $v_3$.

Example. In the real vector space $P$, let the vectors $v = -x^3 - 2x^2 + x + 1, v_1 = x^3 + 3x - 1, v_2 = -x^3 - x^2 - x + 1$ and $v_3 = -5x^3 - 4x^2 - 7x + 5$.

The vector $v$ is a linear combination of the vectors $v_1, v_2$ and $v_3$?

The vector $v$ is a linear combination of the vectors $v_1, v_2$ and $v_3$ if and only if the equation

$v = [\lambda_1 v_1] + \lambda_2 [v_2] + [\lambda_3 v_3]$, with unknown $\lambda_i$, $i = 1, 3$, is possible (fixed or indefinite).

$v = [\lambda_1 v_1] + \lambda_2 [v_2] + [\lambda_3 v_3] \iff [1 -1 -5 0 -1 -4 3 -1 1 -7 5] [\lambda_1 \lambda_2 \lambda_3] = [-1 -2 1 1] \iff \{\lambda_1 = 1 + \beta, \lambda_2 = 2 - 4 \beta, \lambda_3 = \beta \in \mathbb{R}\}$.

then, $v$ is a linear combination of $v_1, v_2$ and $v_3$, in this case, there exist infinite linear combinations of $v_1, v_2, v_3$, equal to $v$.

Definition (Trivial zero linear combination). In a real vector space $V$, the zero vector, $0$, is a linear combination of any vectors system, just take all the same zero scalar in the linear combination. All this zero linear combination are called Trivial zero linear combination.
Definition (Vectors of System Equivalence). Let $V$ be a real vector space. Let $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ and $S'=\{\vec{v}'_1, \vec{v}'_2, \ldots, \vec{v}'_r\}$ two vector systems contained in $V$. It is said that the system of vector $S$ is equivalent to the system of vectors $S'$, and written $S\approx S'$, if $\forall i \in \{1, 2, \ldots, p\}$, $\vec{v}_i$ is linear combination of the vectors of $S'$, and $\forall j \in \{1, 2, \ldots, r\}$, $\vec{v}'_j$ is a linear combination of vectors of $S$.

Theorem (Properties of vector of system equivalence). Let $S$, $S'$ and $S''$ be a system of vectors contained in a real vector space $V$.

1. $S\approx S$;
2. If $S\approx S'$, then $S'\approx S$;
3. If $S\approx S'$ and $S'=S''$, then $S\approx S''$;
4. $\{\vec{v}'_1, \vec{v}'_2, \ldots, \vec{v}'_i, \ldots, \vec{v}'_p\}\approx\{\vec{v}''_1, \vec{v}''_2, \ldots, \vec{v}''_i, \ldots, \vec{v''}_p\}$, where $\lambda \in \mathbb{R}\setminus\{0\}$;
5. $\{\vec{v}'_1, \vec{v}'_2, \ldots, \vec{v}'_i, \ldots, \vec{v}''_j, \ldots, \vec{v}'_p\}\approx\{\vec{v}''_1, \vec{v}''_2, \ldots, \vec{v}''_i, \ldots, \vec{v}''_j, \ldots\}$, where $\lambda \in \mathbb{R}$;
6. $\vec{v}'_i \approx \lambda \vec{v}_i$, where $\lambda \in \mathbb{R}$;
7. If $S\approx S'$ and $\vec{v}$ is a linear combination of the vectors of $S$, then $\vec{v}$ is a linear combination of $S'$.

Definition (Generator of a vector space). Let $V$ be a real vector space. Let $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subset V$. is said that the real vector space $V$ is generated by $S$, or the vectors of $S$, and written as $V=\langle S \rangle$ or $V=\langle \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \rangle$, if for any vector $\vec{v} \in V$ is a linear combination of vectors of $S$.

Example. In the vector space $\mathbb{R}^3$, let the vectors $\vec{v}_1=(1,2,1), \vec{v}_2=(2,1,1), \vec{v}_3=(1,-1,0)$. $\mathbb{R}^3$ is generated by $\vec{v}_1, \vec{v}_2, \vec{v}_3$? $\mathbb{R}^3=\langle \vec{v}_1, \vec{v}_2, \vec{v}_3 \rangle$ if and only if, the equation $\lambda_1 \vec{v}_1+\lambda_2 \vec{v}_2+\lambda_3 \vec{v}_3=(x,y,z)$, with unknowns $\lambda_1, \lambda_2, \lambda_3$, is possible for all vector $(x,y,z) \in \mathbb{R}^3$.

That is, the vectors $(x,y,z) \in \mathbb{R}^3$ that are linear combination of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, are those that satisfied the condition $y+x-3z=0 \iff x=-y+3z$, that is, vectors $(-y+3z,y,z)$, for all real value $y$ and $z$. For example, the vectors $u=(\beta,1,2)$, for any real value $\beta \neq 5$, are not linear combination of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Thus, $\mathbb{R}^3$ is not generated by those vectors.
Example In the vector space $\mathbb{R}^3$, let the vectors

$$v_1=(1,1,-2), v_2=(1,1,1), v_3=(-1,1,1).$$

$\mathbb{R}^3$ is generated by $v_1,v_2,v_3$? $\mathbb{R}^3=(v_1,v_2,v_3)$ if and only if, the equation

$$\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3=(x,y,z),$$

with unknown $\lambda_1, \lambda_2, \text{ and } \lambda_3$, is possible for all vector $(x,y,z) \in \mathbb{R}^3$.

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix} \iff \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Note that the system is fixed for all real value of $x, y \text{ and } z$, that is, the vectors $\mathbb{R}^3=(v_1,v_2,v_3)$.

Theorem. Let $V$ be a real vector space. Then:

1. If $V=(v_1,v_2,...,v_k)$ and $\{v_1,v_2,...,v_k\} \approx \{w_1,w_2,...,w_r\}$, then $V=(w_1,w_2,...,w_r)$.

2. If $V=(v_1,v_2,...,v_k)=(w_1,w_2,...,w_r)$, then $\{v_1,v_2,...,v_k\} \approx \{w_1,w_2,...,w_r\}$.

Definition (Finitely Generate Space). A real vector space $V$ is said to be finitely generated, if there exist a finite system of vectors, $v_1,v_2,...,v_k$, such that $V=(v_1,v_2,...,v_k)$.

Example. The real vector spaces $\mathbb{R}^n, n \in \mathbb{N}$ and $n>1$, and $P^n, n \in \mathbb{N}_0$. However, the real vector space $P$, is not finitely generated.

Definition (Linear Dependent and Independent). Let $V$ be a real vector space. It is said that:

1. The vectors $v_1,v_2,...,v_k \in V$ are linearly independent if

$$\lambda_1 v_1+\lambda_2 v_2+\cdots+\lambda_k v_k=0 \iff \lambda_1=\lambda_2=\cdots=\lambda_k=0,$$

that is, single zero linear combination of $v_1,v_2,...,v_k$ is the trivial zero linear combination.

2. The vectors $v_1,v_2,...,v_k \in V$ are linearly dependent if there exist a real scalar $\lambda_1,\lambda_2,...,\lambda_k$, not all zero, such that $\lambda_1 v_1+\lambda_2 v_2+\cdots+\lambda_k v_k=0$, that is, beyond the trivial zero linear combination, there exist another linear combination of $v_1,v_2,...,v_k$.

Example. In the real vector space $\mathbb{R}^3$, let the vectors

$$v_1=(1,1,1), v_2=(-1,2,1), \text{ and } v_3=(1,-1,1).$$

Show that the vectors $v_1,v_2$ and $v_3$ are linear dependent or independent.

However, the vectors $v_1,v_2$ and $v_3$ are linear dependent (independent) if the equation

$$\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3=0 \iff \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \lambda_1=0$$
Thus, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent.

Note (Another way of solving the previous example). Note that the previous equation can be determined, that is, it has a unique solution, if

$$|A| = |1 -2 1 1 1 -1 1 1 1| \neq 0;$$

Otherwise, it would be likely undetermined. As $|A| = 6$, then the system is likely determined and thus the vectors are linearly independent.

Example. In the real vector space $\mathbb{R}$, let the following vectors

$\mathbf{v}_1 = x^2 + x + 1, \mathbf{v}_2 = -2x^2 + x + 1, \mathbf{v}_3 = 3x + 3$.

Show that these vectors are linearly dependent or independent.

However,

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 = 0 \iff [1 -2 0 1 1 1 3 1 1 3] \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \lambda_1 = -2\lambda, \lambda_2 = -\lambda, \lambda_3 = \lambda \in \mathbb{R},$$

that is, the equation is likely undetermined, then the vectors are linearly dependent.

Note. As

$$|1 -2 0 1 1 3 1 1 3| = 0,$$

it can be concluded that the vector $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.

Theorem (Properties of Linear Dependent and Linear Independent). Let $V$ be a real vector space. Then:

1. A vector $\mathbf{v} \in V$ is linearly dependent if and only if $\mathbf{v} = 0$;
2. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$ are linearly dependent if and only if one of them is a linear combination of the others.
3. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$ are linearly dependent, then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \lambda_1 \mathbf{v}_{(k+1)}, \mathbf{v}_{(k+2)}, \ldots, \mathbf{v}_{(k+t)} \in V,$ $t \in \mathbb{N}$, are also linearly dependent.
4. Any subsystem of a linearly independent system is linearly dependent.
5. Any system of vector that the zero vector is linearly dependent.
6. If the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \lambda \mathbf{v} \in V$ are linearly dependent if and only if, $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.
7. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \lambda \mathbf{v}, \mathbf{v}_i, \ldots, \mathbf{v}_p$, where $\lambda \in \mathbb{R}\{0\}$, are linearly independent.
8. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_p$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \lambda \mathbf{v}, \mathbf{v}_i, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_p$, where $\lambda \in \mathbb{R}$, are also linearly independent.
9. The vectors $v_1, v_2, \ldots, v_k \in V$ are linearly independent if and only if any vector $v \in V$, which can be expressed as a linear combination of them and has unique coefficient.

### Rank of a matrix / Linear Dependent and Independent

Let $A$ be a real matrix of dimension $m \times n$, where $m, n \in \mathbb{N}$. Each row of the matrix $A$ can be seen as a vector of $\mathbb{R}^n$, it is called row vector, and each column of the matrix can be seen as a vector of $\mathbb{R}^m$, is called column vector. Thus,

1. In this set of row vectors of $A$, the number of linearly independent row vectors is equal to the rank of $A$. The linearly independent row vectors are those which correspond to nonzero row of the echelon matrix of $A$.

2. The set of column vectors of $A$, the number of linearly independent column vectors is equal to the rank of $A$. The linearly independent column vector are those which correspond to nonzero column of the echelon matrix obtained from $A$.

**Definition (Maximal Independent Subsystem).** Let $V$ be a real vector space and let $S$ be a system of vector, not all zero, of $V$. A subsystem $S_m$ is said to be maximal independent if it is independent, but become dependent if you add any other vector of $S$.

**Theorem.** Let $V$ be a real vector space and let $S$ be a system of vectors, not all nonzero, of $V$. then, there exist a maximal independent subsystem $S_m$ of $S$. Moreover, all maximal independent subsystem of $S$ has the same number of vectors.

**Theorem.** Let $S = \{v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_{k+t}\}$ be a system of vectors, of a real vector space $V$. Suppose that $S_m = \{v_1, v_2, \ldots, v_k\}$ is a maximal independent subsystem of $S$. Then, $S_m = S$.

**Note.** It is clear that if $S$ is linearly independent, there exist one and only independent subsystem $S_m$ of $S$, which is $S$.

**Example.** In the real vector space $\mathbb{R}^3$, let the vectors $v_1 = (1,1,1), v_2 = (-2,1,1), v_3 = (-3,3,3)$.

Determine a maximal independent subsystem $S_m$ of $S = \{v_1, v_2, v_3\}$.

Let the matrix

$$A = \begin{bmatrix} 1 & -2 & -3 & 1 & 1 & 3 & 1 & 1 & 3 \end{bmatrix},$$

whose columns are the vectors of components $v_i$, $i = 1,3$. The rank of $A$ is the number of vector of any maximal independent subsystem of $S$, and echelon form of $A$ allows to calculate a maximal independent subsystem $S$.

Now,

$$A = \begin{bmatrix} 1 & -2 & -3 & 1 & 1 & 3 & 1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

then, $S_m = \{v_1, v_2\}$.

**Example.** In the real vector space $P$, let the following vectors be $v_1 = x^2 - 1, v_2 = x^2 + 2x + 1, v_3 = x + 1, v_4 = -3x^2 - x + 2$.
Determine a maximal independent subsystem $S_m$ of $S = \{v_1, v_2, v_3, v_4\}$.

Let the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 1 & -1 & 1 & 2 \end{bmatrix},$$

whose columns are the coefficients of the vectors (polynomial) $v_i$, $i = 1, 4$.

Now,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 1 & -1 & 1 & 1 & -3 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/2 & 0 & 1 & 1/2 & 0 & 0 & 0 & -5/2 & -1/2 & 0 \end{bmatrix},$$

Then, $S_m = \{v_1, v_2\}$.

**Theorem of Steinitz**

Lemma (Theorem of Steinitz). Let $V$ be a real vector space and let

$$u = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{i-1} v_{i-1} + \lambda_i v_i + \lambda_{i+1} v_{i+1} + \cdots + \lambda_k v_k,$$

with $u \neq 0$. If

$$u = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{i-1} v_{i-1} + \lambda_i v_i + \lambda_{i+1} v_{i+1} + \cdots + \lambda_k v_k,$$

with $\lambda_i \neq 0$, for any $i \in \{1, 2, \ldots, k\}$.

Theorem (Of Steinitz). Let $u_1, u_2, \ldots, u_p$ be linearly independent of a real vector space $V$. Suppose that each of the vectors $u_i$, $i = 1, p$ is a linear combination of certain $v_1, v_2, \ldots, v_k$ of $V$. Then, you can replace $p$ of the vectors $v_1, v_2, \ldots, v_k$ by vectors $u_1, u_2, \ldots, u_\nu$, and obtain an equivalent system $v_1, v_2, \ldots, v_k$.

**Example (Application of the Theorem of Steinitz).** In the real vector space $\mathbb{R}^3$, let the vectors

$$u_1 = (6, -2, 1), u_2 = (-1, 4, 4), v_1 = (1, 1, 2), v_2 = (-2, 1, 1), v_3 = (1, -1, 1).$$

a. Show that if you replace 2 vectors in the system $S = \{v_1, v_2, v_3\}$ by the vectors $u_1$ and $u_2$, then find the system $S'$ equivalent to $S$.

b. If obtain, determine $S'$.

**Paragraph (a).** However, $u_1$ and $u_2$ must be linearly independent (and are or are not proportional), and each of them has to be a linear combination of the vectors $v_i$, $i = 1, 3$, that is,

$$u_1 = [\lambda_1 v_1]_1 + \lambda_2 v_2 + [\lambda_3 v_3]_3 \quad \text{and} \quad u_2 = [\beta_1 v_1]_1 + \beta_2 v_2 + [\beta_3 v_3]_3,$$

For some real scalar $\lambda_i, \beta_i, i = 1, 3$. 

$$\begin{bmatrix} A | B_1 B_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & 6 & -1 \\ 1 & 1 & -1 & -2 & 4 \\ 2 & 1 & 1 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$
That is, \( \vec{u}_{1} = \vec{v}_{1} - 2\vec{v}_{2} + \vec{v}_{3} \) and \( \vec{u}_{2} = [2\vec{v}]_{1} + \vec{v}_{2} - \vec{v}_{3} \).

Paragraph (b). By the Lemma from Teorema of Steinitz, \( S \approx S_{1} = \{\vec{u}_{1}, \vec{v}_{2}, \vec{v}_{3}\} \).

As \( \vec{v}_{1} \) is replaced by \( \vec{u}_{1} \), we have:

\[
\vec{u}_{1} + 2\vec{v}_{2} - \vec{v}_{3} = \vec{v}_{1} ,
\]

Now

\[
\vec{u}_{2} = 2(\vec{u}_{1} + 2\vec{v}_{2} - \vec{v}_{3}) + \vec{v}_{2} - \vec{v}_{3} = 2\vec{u}_{1} + 3\vec{v}_{2} - 2\vec{v}_{3} .
\]

By the Lemma from Theorem of Steinitz, \( S_{1} \approx S^{'} = \{\vec{u}_{1}, \vec{v}_{2}, \vec{u}_{2}\} \). As \( S \approx S_{1} \) and \( S_{1} \approx S^{'} \), then \( S \approx S^{'} \).

Consequence of Theorem of Steinitz

Corollary 1. In a real vector space, \( p \) linearly independent vectors can be written as linear combination of any \( k \) vectors, then \( p \leq k \).

Corollary 2. In a real vector space \( V \), two equivalent of vectors system, both linearly independent, have the same number of vectors.

Corollary 3. In the real vector space \( V \), two equivalent of vectors system, with the same number of vector of the same type ie dependent or independent.

Basis and dimension of a real vector space

Definition (Basis of a vector space finitely generated). A real nonzero vector space finitely generated is called system of linearly independent generator.

By convention, the zero vector \( \mathbb{V} = \{0\} \) as the basis of the empty set \( \emptyset \).

Note. By convention, a basis vectors are considered written in a certain order, so that the same vectors for another order, an order different from the first.

Theorem. All real vector space \( V \), finitely generated, has at least one basis.

Theorem. two basis of the same real vector space \( V \), finitely generated has the same number of vectors.

Definition (Dimension of vector space finitely generated). It is called the dimension of a real vector space \( V \), finitely generated, the number of vector of any of its basis and is written as \( \dim(V) \).

A real vector space finitely generated is said to be finite dimension while a real vector space not finitely generated is said to be infinite dimension..

Theorem (Dimension / Basis). Let \( V \) be a real vector space of dimension \( n \in \mathbb{N} \). Then:

1. Any system of \( n \) vectors, of \( V \), linearly independent, form a basis.
2. Any system of \( n \) generators of \( V \) form a basis.
3. Any system with more than \( n \) vectors is linearly independent
4. If \( \vec{v}_{1}, \vec{v}_{2}, \ldots, \vec{v}_{k} \in V \) are linearly independent, then one can construct a basis of \( V \) that
include these vectors (just use the Theorem of Steinitz, for any basis of V).

Definition (Dimension of a real vector space finitely generated). Let \( B = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) be a basis of the real vector space V. then, each vector \( \vec{v} \in V \) written in unique way as a linear combination of the vector of B, that is, there exist a unique real scalar \( \lambda_1, \lambda_2, \ldots, \lambda_k \) such that

\[
\vec{v} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \cdots + \lambda_k \vec{v}_k
\]

In this case, the unique real scalar \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are called components of the vector \( \vec{v} \) in basis B, and is written as

\[
\vec{v} \equiv_B (\lambda_1, \lambda_2, \ldots, \lambda_k)
\]

Note. The components of a vector, in a real vector space of dimension k, is a list of k real numbers.

Components of a vector and the eigenvector are different. A vector is a well-define element of a real vector space V, while its component is a sequence of real number whose value depends on the considered basis.

Canonical Basis of \( \mathbb{R}^n \), \( n \in \mathbb{N} \) and \( n>1 \).

\( B = \{ (1,0,0,\ldots,0), (0,1,0,\ldots,0), (0,0,1,\ldots,0), \ldots, (0,0,0,\ldots,1) \} \) is a canonical basis \( \mathbb{R}^n \). Therefore \( \dim(\mathbb{R}^n)=n \).

Canonical Basis of \( \mathbb{P}^n \), \( n \in \mathbb{N}_0 \).

\( B = \{ x^n, x^{n-1}, x^{n-2}, \ldots, x, 1 \} \) is a canonical basis of \( \mathbb{P}^n \). Therefore \( \dim(\mathbb{P}^n)=n+1 \).

Example. Show that if a set \( B = \{ \vec{v}_1=(1,-1,2), \vec{v}_2=(0,1,1), \vec{v}_3=(2,-2,1) \} \) is a basis of the real vector space \( \mathbb{R}^3 \).

As \( \dim(\mathbb{R}^3)=3 \), then B is a basis of \( \mathbb{R}^3 \) is linearly independent, that is, the rank of the matrix

\[
A = \begin{bmatrix}
1 & 0 & 2 \\
-1 & 1 & -2 \\
2 & 1 & 1
\end{bmatrix}
\]

\( c(A)=3 \) if and only if, \( |A|\neq0 \). \( |A|=-3 \), then B is a basis of \( \mathbb{R}^3 \).

Example. Determine the component of the vector \( \vec{v}=(x,y,z) \in \mathbb{R}^3 \), based on basis B don the previous example.

Now

\[
\vec{v}=(x,y,z)=\lambda_1 (1,-1,2)+\lambda_2 (0,1,1)+\lambda_3 (2,-2,1),
\]

for some real scalar \( \lambda_1, \lambda_2, \lambda_3 \).

\[
\begin{bmatrix}
1 & 0 & 2 \\
-1 & 1 & -2 \\
2 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix}
= \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\iff
\begin{cases}
\lambda_1 = (\frac{-3x - 2y + 2z}{3}) \\
\lambda_2 = x + y \\
\lambda_3 = (\frac{3x + y - z}{3})
\end{cases}
\]
Then
\[ \vec{v} \equiv _B ((-3x-2y+2z)/3,x+y,(3x+y-z)/3). \]

Example. In the real vector space \( \mathbb{P}^2 \), determine the components of vector \( \vec{v} = -3x^2-x-2 \) with respect to the basis \( B = \{x^2,x-1,-x^2+2\} \).

Now \( \vec{v} = -3x^2-x-2 = \lambda_1 x^2 + \lambda_2 (x-1) + \lambda_3 (-x^2+2) \),
for some real scalar \( \lambda_1, \lambda_2, \lambda_3 \).

Then,
\[ \vec{v} \equiv _B ((-9)/2,-1,(-3)/2). \]

Example. In the real vector space \( \mathbb{R}^3 \), let the vectors \( \vec{v}_1 = (1,1,\lambda), \vec{v}_2 = (0,1,-\lambda) \) and \( \vec{v}_3 = (\lambda,1,-1) \). Determine the real value of \( \lambda \) such that \( B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \) is a basis of \( \mathbb{R}^3 \).

Let the real matrix
\[ A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 2 \end{bmatrix} \]

\( B \) is a basis if and only if the vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are linearly independent. But, \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are linearly independent if and only if, \( |A| \neq 0 \).

\( |A| \neq 0 \iff -\lambda^2 + \lambda - 1 \neq 0. \) Como \(-\lambda^2 + \lambda - 1 < 0\), for all \( \lambda \in \mathbb{R} \), then \( B \) is a basis of \( \mathbb{R}^3 \), for all \( \lambda \in \mathbb{R} \).

**Conclusion**

Knowledge of properties of vectors in a real vector space is essential.

The concept of linear combination, the real vector space, giving rise to other fundamental concepts such as linear dependence and independence generator and, consequently, basis and dimension.

**Assessment**

(1) Let \( \vec{v}_1, \vec{v}_2 \) and \( \vec{v}_3 \) linearly independent vectors of a real vector space \( V \). What can be said about the linear dependence and linear independence of the system \( S = \{\vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_3, \vec{v}_2 + \vec{v}_3\} \).

(2) Find the value of parameter \( \alpha \), the set \( B = \{(1,1,\alpha), (\alpha,1,1), (-1,1,\alpha)\} \) is a basis of the real vector space \( \mathbb{R}^3 \).
In the real vector space $\mathbb{P}^3$, consider the basis $B=\{x^3-x^2,x^2-x,x-1,-x+2\}$.

(a) Justify if it is possible to obtain a basis of the real vector space $\mathbb{P}^3$, including the vectors $\vec{u} = x^3-x^2+x+1$, $\vec{w} = x^3+x-1$.

(b) Apply the Theorem of Steinitz for such a basis.

(4) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and $\vec{v}_4$ linearly independent vectors of a real vector space $V$. Determine a basis and the dimension of the real vector space $X = \langle \vec{v}_1+\vec{v}_2, \vec{v}_2-2\vec{v}_4, \vec{v}_1+2\vec{v}_4, \vec{v}_4 \rangle$.

(5) In the real vector space $\mathbb{R}^3$, determine the components of the vector with respect to the basis $B=(1,1,1),(-1,0,1),(-2,1,2)$.

Summary

In this unit, we introduced the concept of space and real vector subspace, highlighting the spaces (or subspaces) of finite dimensions.

The knowledge basis and size of a given real vector space, allows us to identify with rigor, any vector included in it.

The inner product is introduced into a real vector space, a fact which has, among other things, determining the norm of a vector calculation of the angle between vectors, standardization of a base, orthogonal projection of calculation of a vector and onto a subspace vector.

Unit Assessment

Check your understanding!

Summary Test on vector space and vector subspace

Instructions

The evaluation test has seven questions, with some points.

Answer each question clearly and justify each step of solution.
Unit 2. Real Vector Space

Grading Scheme

Each question worth 10 points. It is considered passed the student who has at least 50% of the total.

Feedback

1. In the real vector space $P^3$, let
   
   $B=\{x^3-x^2+1,x^2-x-2,(k-1)x^3+x^2+x-1,-x^2+x+3\}, k \in \mathbb{R}$.

   Determine and show the value of the real parameter $k$, such that $B$ is a basis of $P^3$.

2. In the vector space $\mathbb{R}^4$, let the following vector subspace be:
   
   $F=\{(1,0,1,-1)\}, G=\{(1,-1,1,1),(0,1,-1,1)\}$ and $H=\{(1,1,-1,3),(1,-1,1,0)\}$.

   Show that $F+G+H=F \oplus G \oplus H$.

3. In the real vector space $P^3$, let the vector subspace $S=\{p(x) \in P^3 : p(1)=0\}$.

   Show that $S \subseteq P^3$. Determine a basis of dimension of $S$.

4. In the real vector space $P^2$, let the following vector subspace be:
   
   $F=\{a(x^2-x)+b(x-1)+c(-x^2+2x) : a=b-c=0\}$ and
   
   $G=\{ax^2+bx+c : a-b=a+b+c=0\}$.

   Determine $F \cap G$.

5. In the real vector space $\mathbb{R}^3$, let the following inner product be:
   
   $\forall u \overset{\rightharpoonup}{=}=(u_1,u_2,u_3), v \overset{\rightharpoonup}{=}=(v_1,v_2,v_3), \quad u \overset{\rightharpoonup}{\cdot} v \overset{\rightharpoonup}{=}=2u_1 v_1+u_2 v_1-u_3 v_1+u_1 v_2+3u_2 v_2-u_1 v_3+u_3 v_3$.

   a. Show that the real values of $\alpha$ are orthogonal to vectors $a \overset{\rightharpoonup}{=}=(2,\alpha,1)$ and $b \overset{\rightharpoonup}{=}=(\alpha+1,2,-1)$.

   b. Determine a unique and orthogonal vector $u \overset{\rightharpoonup}{=}=(1,-1,2)$ and $v \overset{\rightharpoonup}{=}=(2,1,-1)$.

6. In the euclidean space $\mathbb{R}^3$, consider the following inner product:
   
   $\forall u \overset{\rightharpoonup}{=}=(u_1,u_2,u_3), v \overset{\rightharpoonup}{=}=(v_1,v_2,v_3) \in \mathbb{R}^3,$

   $u \overset{\rightharpoonup}{\cdot} v \overset{\rightharpoonup}{=}=2u_1 v_1+u_2 v_1-u_3 v_1+u_1 v_2+3u_2 v_2-u_1 v_3+u_3 v_3$.

   Use the Gram-Schmidt process of orthogonalization to the basis $B=\{(1,1,1),(1,0,1),(1,2,3)\}$ to an orthonormal basis.

7. In the euclidean space $\mathbb{R}^3$, with the canonical inner product, let the vectors

   $u \overset{\rightharpoonup}{=}=(-1,1,2), v \overset{\rightharpoonup}{=}=(0,-1,2)$.

   Determine the area of the parallelogram that whose non-parallel sides are vectors $u \overset{\rightharpoonup}{\rightharpoonup}$ and $v \overset{\rightharpoonup}{\rightharpoonup}$. 

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Unit Readings and Other Resources

The readings in this unit are to be found at course level readings and other resources.

- LAY, D.C., Linear Algebra and its Application, Addison-Wesley Production Company, 1994
- [https://en.wikipedia.org/wiki/Linear_algebra](https://en.wikipedia.org/wiki/Linear_algebra)
Unit 3. Linear Transformation

Unit Introduction

This unit will study the linear transformation of an arbitrary vector space $V$ to another arbitrary vector space $V'$. The results we are going to get have various applications in Physics, Engineering and various branches of mathematics. It can also be used in computer science that is computer vision, graphic, machine learning, etc.

Unit Objectives

Upon completion of this unit, you should be able to:

- Identify a linear application.
- Determining the core and the image of a linear transformation.
- Identify an isomorphism.
- Determine matrix of a linear transformation with respect to a certain basis.
- Operate with linear applications.

Learning Activities

Activity 3.1 - Linear Application: kernel and image

Introduction

This activity introduces the concept of linear transformation, and gives up the main emphasis on the determination of the core image.

The terms “image” and “inverse image” in a subspace or a vector as well as their properties are fundamental in determining the core and the image and hence the characterization of a linear transformation (monomorphic, epimorphism and isomorphism).

It shows the relationship between the size of the starting space of a linear transformation, with the nullity (size core) and feature (image size).
**Key Terms**

Linear Application: Let $V$ and $V'$ be vector space on a field $R$. An application $f: V \longrightarrow V'$ is said to be a linear application, if $\forall \vec{u}, \vec{v} \in V, \lambda \in R$ we have:

$$f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v});$$

$$f(\lambda \vec{u}) = \lambda f(\vec{u}).$$

Image and inverse image of a set: Let $f: V \longrightarrow V'$ be a linear application, between the real vector space $V$ and $V'$. Let $F \subseteq V$ and $F' \subseteq V'$. This image of $F$ is

$$f(F) = \{f(\vec{v}) : \vec{v} \in F\},$$

and the inverse image of $F'$ is

$$f^\leftarrow(F') = \{\vec{v} \in V : f(\vec{v}) \in F'\}.$$  

Linear Transformation of a Matrix: Let $f: V \longrightarrow V'$ be a linear transformation between the vector space $V$ and $V'$ in the field $R$, with dimension $n$ and $p$, respectively. Let

\[ B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \quad \text{and} \quad B' = \{\vec{v}'_1, \vec{v}'_2, \ldots, \vec{v}'_p\} \]

be arbitrary basis of $V$ and $V'$, respectively. Let Sejam

\[
\begin{align*}
  f(\vec{v}_1) & \equiv \_B' (a_{11}, a_{21}, \ldots, a_{p1}) \\
  f(\vec{v}_2) & \equiv \_B' (a_{12}, a_{22}, \ldots, a_{p2}) \\
  \vdots & \vdots \vdots \\
  f(\vec{v}_n) & \equiv \_B' (a_{1n}, a_{2n}, \ldots, a_{pn})
\end{align*}
\]

Then, the linear transformation of the matrix $f: V \longrightarrow V'$ with respect to the basis $B$ and $B'$ is

\[
A = M(f; B, B') = [a_{11} \quad a_{12} \ldots a_{21} \quad a_{22} \ldots : a_{p1} \quad a_{p2} : \ldots \quad a_{1n} \quad a_{2n} : a_{pn}] ,
\]

that is, it is the matrix of dimension $p \times n$, whose column $i$ is composed of components of vector $f(\vec{v}_i)$ in the basis $B'$. 

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Activity Details

Definition (Linear Transformation). Let V and V’ be vector spaces in any body R. An applicationação

\[ f: V \rightarrow V' \]

it is called a linear transformation, if \( \forall u, v \in V, \lambda \in \mathbb{R} \) we have:

\[ f(u + v) = f(u) + f(v) ; \]

\[ f(\lambda u) = \lambda f(u) . \]

Theorem. The above definition is equivalent to the following: \( f: V \rightarrow V' \) is a linear transformation if \( \forall u, v \in V, \lambda, \beta \in \mathbb{R} \) we have:

\[ f(\lambda u + \beta v) = \lambda f(u) + \beta f(v) . \]

Example. The transformation \( f: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), defined by

\[ f(x, y, z) = (3x - 7z, x - 5y + 8z) , \]

is a linear transformation.

Proof. Let \( u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3 \) and let \( \lambda \in \mathbb{R} . \)

\[ f(u + v) = f(u_1 + v_1, u_2 + v_2, u_3 + v_3) \]

\[ = (3(u_1 + v_1) - 7(u_3 + v_3), u_1 + v_1 - 5(u_2 + v_2) + 8(u_3 + v_3)) \]

\[ f(u) + f(v) = (3u_1 - 7u_3, u_1 - 5u_2 + 8u_3) + (3v_1 - 7v_3, v_1 - 5v_2 + 8v_3) \]

\[ = (3(u_1 + v_1) - 7(u_3 + v_3), u_1 + v_1 - 5(u_2 + v_2) + 8(u_3 + v_3)) \]

That is, \( f(u + v) = f(u) + f(v) . \)

\[ f(\lambda u) = f(\lambda u_1, \lambda u_2, \lambda u_3) \]

\[ = (3(\lambda u_1) - 7(\lambda u_3), \lambda u_1 - 5(\lambda u_2) + 8(\lambda u_3)) \]

\[ \lambda f(u) = \lambda (3u_1 - 7u_3, u_1 - 5u_2 + 8u_3) \]

\[ = ((\lambda 3u_1) - 7(\lambda u_3), \lambda u_1 - 5(\lambda u_2) + 8(\lambda u_3)) \]

That is, \( f(\lambda u) = \lambda f(u) . \)

Null Linear Transformation. Let V and V’ be a vector spaces on any body R. The linear transformation

\[ 0: V \rightarrow V' \]

such that \( \forall v \in V, 0(v) = 0' \), is called null linear transformation.

Linear Transformation Identity. Let V be a vector space over any body R. The linear transformation

\[ i: V \rightarrow V \]

such that \( \forall v \in V, i(v) = v \), is called linear transformation identity.
Note. We can define a linear transformation $f: V \rightarrow V'$, between vector space $V$ and $V'$ on any body $R$, and to publish the images of all the vectors of any basis $V$.

Example. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation defined by:

$$f((1,1)) = (3,2,1) \text{ and } f((0,-2)) = (0,1,0).$$

Determine $f((a,b))$, for all $(a,b) \in \mathbb{R}^2$.

As the vectors $(1,1)$ and $(0,-2)$ is a base of $\mathbb{R}^2$, then any vector $(a,b) \in \mathbb{R}^2$, is a linear combination of these vectors, that is, there exist scalars $\lambda_1$ e $\lambda_2$ such that

$$(a,b) = \lambda_1 (1,1) + \lambda_2 (0,-2).$$

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ a & b & -1 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ a & b & -a \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ a & (a-b)/2 \end{bmatrix}$$

Then, $\lambda_1 = a$ e $\lambda_2 = (a-b)/2$.

Therefore,

$$f((a,b)) = f(a(1,1)+(a-b)/2(0,-2))$$

$$= af((1,1))+(a-b)/2 f((0,-2))$$

$$= a(3,2,1)+(a-b)/2(0,1,0)$$

$$= (3a,(5a-b)/2,a)$$

A linear transformation $f: V \rightarrow V'$, between vector spaces $V$ and $V'$ on any body $R$, it is said:

Monomorphism, if it is injective.

Epimorphism, if it is subjective.

Isomorphism, if it is both monomorphism and epimorphism .

Endomorphism , if $V = V'$.

Automorphism, if it is both isomorphism and endomorphism.

Theorem. Let $f: V \rightarrow V'$ a linear transformation between vector spaces $V$ and $V'$ on the body $R$. Then:

$$f((0_V) \rightarrow (0_{V'}), \text{ where } (0_V) \rightarrow \text{ and } (0_{V'}) \rightarrow \text{ are null vectors } V \text{ and } V', \text{ respectively.}$$

$$\forall v \in V, f(-v) = -f(v).$$

$$\forall (u,v) \in V, f(u-v) = f(u) - f(v).$$

$f$ becomes linearly dependent vectors of $V$ in linearly dependent vectors $V'$, that is, if $(v_1, v_2, ..., v_n)$ are linearly dependent vectors of $V$, then $f((v_1), f((v_2)), \ldots, f((v_n))$ are also linearly dependent.

$f$ becomes linearly independent vector of $V$ in linearly independent vectors of $V'$ if and only if $f$ is injective.
Unit 3. Linear Transformation

Definition (Image and inverse image). Let $f: V \rightarrow V'$ a linear transformation between the real vector spaces $V$ and $V'$. Let $F \subseteq V$ and $F' \subseteq V'$. Image of $F$ is

$$f(F) = \{ f(\vec{v}) : \vec{v} \in F \},$$

and inverse image of $F'$ is

$$f^{-1}(F') = \{ \vec{v} \in V : f(\vec{v}) \in F' \}.$$

Note. $f(\{ \vec{v} \})$ and $f^{-1}(\{ \vec{v} \}) = f^{-1}(\vec{v})$.  

Example. Let the linear transformation $f: P^2 \rightarrow P^1$, defined by

$$f(ax^2 + bx + c) = (a - c)x + b,$$

for all $a, b, c \in \mathbb{R}$.

Let $F = \{ x, x - 1 \}$, $G = \langle x, x - 1 \rangle$, $F^\prime = \{ x + 1, -x \}$ and $G^\prime = \langle -2x + 1 \rangle$. Determine $f(F)$, $f(G)$, $f^{-1}(F')$ and $f^{-1}(G')$.

**Solution:**

1. **Determine $f(F)$**

   $$f(F) = \{ f(x), f(x - 1) \} = \{ 1, x - 1 \}.$$

2. **Determine $f(G)$**

   

   $G = \langle x, x - 1 \rangle = \{ \alpha x + \beta (x - 1) : \alpha, \beta \in \mathbb{R} \} = \{ (\alpha + \beta) x - \beta : \alpha, \beta \in \mathbb{R} \}.$

   Then:

   $$f(G) = \{ f((\alpha + \beta) x - \beta) : \alpha, \beta \in \mathbb{R} \} = \{ \beta x + \alpha + \beta : \alpha, \beta \in \mathbb{R} \} = \{ \beta (x + 1) + \alpha : \alpha, \beta \in \mathbb{R} \} = \langle x + 1, 1 \rangle = P^1.$$

3. **Determine $f^{-1}(F')$**

   

   $$f(ax^2 + bx + c) \in F' \iff f(ax^2 + bx + c) = x + 1 \text{ or } f(ax^2 + bx + c) = -x.$$

   $$\iff (a - c)x + b = x + 1 \text{ or } (a - c)x + b = -x.$$

   $$\iff (a - c = 1 \text{ and } b = 1) \text{ or } (a - c = -1 \text{ and } b = 0).$$

   Then:

   $$f^{-1}(F') = \{ (1 + c)x^2 + x + c : c \in \mathbb{R} \} \cup \{ (-1 + c)x^2 + x + c : c \in \mathbb{R} \} = \langle x^2 + x, x^2 + 1 \rangle.$$  

4. **Determine $f^{-1}(G')$**

   

   $G' = \langle -2x + 1 \rangle = \{ \lambda(-2x + 1) : \lambda \in \mathbb{R} \}$

   $$f^{-1}(G') = \{ ax^2 + bx + c \in P^2 : f(ax^2 + bx + c) = -2\lambda x + \lambda \}.$$

   

   $$f(ax^2 + bx + c) = -2\lambda x \iff +\lambda(a - c)x + b = -2\lambda x + \lambda.$$

   $$\iff a - c = -2\lambda \text{ and } b = \lambda.$$

   Then:

   $$f^{-1}(G') = \{ (c - 2\lambda)x^2 + \lambda x + c : \lambda, c \in \mathbb{R} \} = \langle -2x^2 + x, x^2 + 1 \rangle.$$  

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Theorem. Let \( f: V \rightarrow V' \) the linear transformation between vector spaces \( V \) and \( V' \) on any body \( R \), and let \( F \) and \( F' \) subspace vector of \( V \) and \( V' \) respectively.

Then:

\[
f(F) = \{ f(v) : v \in F \} \text{ (called image of } F \text{)} \text{ is a subspace vectorial of } V'.
\]

\[
f^{-1}(F') = \{ v \in V : f(v) \in V' \} \text{ (called inverse image of } F' \text{)} \text{ is a subspace of } V.
\]

Note. Let \( f: V \rightarrow V' \) a linear transformation between vector spaces \( V \) and \( V' \) on any body \( R \). It is known that the vector space \( V \) is a vector subspace itself, and \( \{0 \} \) is a vector subspace \( V' \), where \( 0 \) is the zero vector in the vector space \( V' \).

Then:

\[
f(V) \text{ is a vector subspace of } V',
\]

\[
f^{-1}(\{0\}) \text{ is a vector subspace of } V.
\]

Definition (Image and Core). Let \( f: V \rightarrow V' \) be a linear transformation between vector spaces \( V \) and \( V' \) on any body \( R \). The vector subspace of \( V' \), \( f(V) = \text{Im}(f) = \text{Im}^0 f \), it is image, or codomain, or space-image or characteristic space of \( f \). When the space-image is finitely generated, its size is called characteristic of \( f \), and represented by \( c_f \), that is, \( c_f = \text{dim}(f(V)) \).

The vector subspace of \( V \), \( f^{-1}(\{0\}) = f^{-1}(0) = \text{Nuc}(f) = \text{Ker}(f) \), it is said core or null space of \( f \). When the core is finitely generated, its size is called a null of \( f \), and represented by \( n_f \), that is, \( n_f = \text{dim}(\text{Nuc}(f)) \).

Example. Let the linear transformation \( f: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \), defined by

\[
f((x,y,z)) = (3x-7z, x-5y+8z),
\]

for all \((x,y,z)\in \mathbb{R}^3\).

Determine the image of \( f \) and \( c_f \), core of \( f \) and \( n_f \).

\[
\text{Im}(f) = f(\mathbb{R}^3) = \{ f((x,y,z)) : (x,y,z) \in \mathbb{R}^3 \}.
\]

Then,

\[
f((x,y,z)) = (3x-7z, x-5y+8z)
\]

\[
= (3x,x) + (0,-5y) + (-7z,8z)
\]

\[
= x(3,1) + y(0,-5) + z(-7,8)
\]

\[
= \left\langle (3,1),(0,-5),(-7,8) \right\rangle
\]

As \( \text{Im}(f) \) is a vector subspace of the vector space \( \mathbb{R}^2 \) and \( \text{dim}(\mathbb{R}^2) = 2 \), then \( c_f \leq 2 \). On the other hand, the system of generators of \( \text{Im}(f) \) two linearly independent vectors, for example, \((3,1)\) e \((0,-5)\), then \( c_f = 2 \).

\[
\text{Nuc}(f) = f^{-1}(\{0\}) = \{ (x,y,z) \in \mathbb{R}^3 : f((x,y,z)) = (0,0) \}.
\]
Now,
\[ f((x,y,z)) = (0,0) \]
\[ \Leftrightarrow \begin{align*} 3x - 7z &= 0 \\ x - 5y + 8z &= 0 \end{align*} \]

Solve this homogeneous system using reducing its extended matrix to reduced stepwise.
\[
\begin{bmatrix} 3 & 0 & -7 & 1 & -5 & 8 \\ 0 & 30 & -7 & 31/3 & 0 & 1 & -31/15 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -7 & 0 & -5 & 31/3 \\ 1 & 0 & (-7)/3 & 0 & 1 & (-31)/15 \end{bmatrix}
\]
It has then:
\[
\begin{align*} x + (-7)/3 z &= 0 \\ y + (-31)/15 z &= 0 \end{align*} \Leftrightarrow \begin{align*} x &= 7/3 z \\ y &= 31/15 z \end{align*} \Leftrightarrow \begin{align*} x &= 7/3 \lambda \\ y &= 31/15 \lambda \\ z &= \lambda \in \mathbb{R} \end{align*}
\[
\text{Nuc}_f = \{(7/3 \lambda, 31/15 \lambda, \lambda) : \lambda \in \mathbb{R}\} = \langle (35, 31, 15) \rangle
\]
Thus, the null of f is equal to one, that is, \( n_f = 1 \).

Theorem. Let \( f: V \to V' \) be a linear transformation between vector space \( V \) and \( V' \) on any body R. Given that vectors \( \vec{v} \in V \) and \( \vec{v}' = f(\vec{v}) \), we have:
\[
f^{-1}(\vec{v}') = \{ \vec{v} \in V : f(\vec{v}) = \vec{v}' \} = \{ \vec{v} \} + \text{Nuc}(f).
\]
\( f \) is a monomorphism if and only if, \( \text{Nuc}(f) = \{0\} \), that is, \( \text{Nuc}(f) \) is the null subspace of \( V \).

Example. The linear transformation \( f: \mathbb{R}^3 \to \mathbb{R}^2 \), defined by
\[
f((x,y,z)) = (3x-7z, x-5y+8z),
\]
for all \((x,y,z) \in \mathbb{R}^3\), has different core null subspace, \( \{0,0,0\} \), of \( \mathbb{R}^3 \), because \( \text{Nuc}_f = \langle (35, 31, 15) \rangle \) (see previous example). Once the transformation is not a monomorphic of \( f \), that is, is not injective.

Example. Let the endomorphism \( f: P^2 \to P^2 \), defined by:
\[
\forall p(x) = ax^2 + bx + c \in P^2, f(p(x)) = p'(x) = 2ax + b.
\]
Determine \( f^{-1}(3x+1) \) and show that \( f \) is a monomorphism.
\[
f^{-1}(3x+1) = \{ ax^2 + bx + c : f(ax^2 + bx + c) = 3x + 1 \}.
\]
\[
f(ax^2 + bx + c) = 3x + 1 \Leftrightarrow 2ax + b = 3x + 1 \]
\[
\Leftrightarrow 2a = 3 \text{ and } b = 1 \text{ and } c \in \mathbb{R}
\]
\[
\Leftrightarrow a = 3/2 \text{ and } b = 1 \text{ and } c \in \mathbb{R}
\]
Then,
\[
f^{-1}(3x+1) = \{ 3/2 x^2 + x + c : c \in \mathbb{R} \} = \{ 3/2 x^2 + x \} + \{ c : c \in \mathbb{R} \} = \{ 3/2 x^2 + x \} + \{0\}
\]
That is, \( \text{Nuc}_f = \langle 1 \rangle \neq \{0\} \), which means that \( f \) is not a monomorphism.

Example. Let the endomorphism \( f: \mathbb{R}^3 \to \mathbb{R}^3 \), defined by
\[
f((x,y,z)) = (x+y-z, y+z, x-y+z), \forall (x,y,z) \in \mathbb{R}^3.
\]
Determine \( f^\leftarrow((-2,1,4)) \) and show that \( f \) is a monomorphism.

\[
f^\leftarrow((-2,1,4)) = \{ (x,y,z) \in \mathbb{R}^3 : f((x,y,z)) = (-2,1,4) \}.
\]

\[
f((x,y,z)) = (-2,1,4)
\]

\[\iff\]

\[
\begin{align*}
(x+y-z, y+z, x-y+z) &= (-2,1,4) \\
(x+y-z) &= -2 \\
y+z &= 1 \\
x-y+z &= 4
\end{align*}
\]

The system is solved using reducing method. It is extended matrix to reduced stepwise.

\[
\begin{bmatrix}
1 & 1 & -1 & 0 & 1 & 1 & -1 & 1 \\
0 & -2 & 1 & 4 & -1 & 1 & 0 & 2 \\
1 & 1 & 1 & 0 & 1 & 0 & 4 & -2 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & -2 & 1 & 4 & -1 & 1 & 0 & 2 & 4 \\
1 & 1 & 1 & 0 & 1 & 0 & 4 & -2 & 6 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 4 \\
0 & -2 & 1 & 4 & -1 & 1 & 0 & 2 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & -2 & 1 & 4 & -1 & 1 & 0 & 2 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\
\end{bmatrix}
\]

Then,

\[
\begin{align*}
(x+y-z) &= -2 & y+z &= 1 & x-y+z &= 4
\end{align*}
\]

Thus, \( f^\leftarrow((-2,1,4)) = \{(-1,1,2)\} \), that is, \([\text{Nuc}]_f = \{0,0,0\}\), therefore, \( f \) is a monomorphism.

**Theorem.** Let \( f : V \rightarrow V' \) a linear transformation between vector space \( V \) and \( V' \) on any body \( R \). If \( F \) is a vector subspace of \( V \) and \( F = \langle X \rangle \), then

\[
f(F) = \langle f(X) \rangle.
\]

In particular, if \( F \) is finitely generated, then \( f(F) \) is also finitely generated, and

\[
dim(f(F)) \leq dim(F).
\]

**Example.** Let the endomorphism \( f : P^2 \rightarrow P^2 \), defined by:

\[
\forall \ p(x) = ax^2 + bx + c \in P^2, f(p(x)) = p'(x) = 2ax + b.
\]

Calculate \( f(F) \), where \( F = \langle x^2 - 1, x \rangle \) is a subspace of \( P^2 \).

By the previous theorem, \( f(F) = \langle f(x^2 - 1), f(x) \rangle = \langle 2x, 1 \rangle = P^1. \)

**Note.** The theorem applies in determining the image of a linear transformation \( f : V \rightarrow V' \), since they meet a base of \( V \).

**Example.** Determine the image of the endomorphism \( f : P^2 \rightarrow P^2 \), defined by:

\[
\forall \ p(x) = ax^2 + bx + c \in P^2, f(p(x)) = p'(x) = 2ax + b.
\]

Since \( P^2 = \langle x^2, x, 1 \rangle \), then

\[
\text{Im}(f) = f(P^2) = \langle f(x^2), f(x), f(1) \rangle = \langle 2x, 1, 0 \rangle = \langle 2x, 1 \rangle = P^1. \)

**Theorem.** Let \( f : V \rightarrow V' \) be a linear transformation between vector spaces \( V \) and \( V' \) on any body \( R \). If the vector space \( V \) has finite dimension, then

\[
dim(V) = c_f + n_f.
\]

In particular, if \( dim(V) = dim(V') = n \), then \( f \) is a monomorphism if and only if \( f \) is an epimorphism.
Unit 3. Linear Transformation

Example. It is known that core endomorphism \( f: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), defined by

\[ f((x,y,z)) = (x+y-z, y+z, x-y+z), \forall (x,y,z) \in \mathbb{R}^3, \]

is a null space \( \{(0,0,0)\} \). Now the null of \( f \) is equal to zero \((0)\). Therefore, it has:

\[ \dim(\mathbb{R}^3) = c_f + n_f = c_f = 3, \]

that is, \( \text{Im}(f) = \mathbb{R}^3, \mathbb{R}^3 \) as any subspace of the same size wherein \( \mathbb{R}^3 \) is equal to \( \mathbb{R}^3 \), consequently \( f \) is an epimorphism. Then, we can say that \( f \) is an automorphism, it is endomorphism and isomorphism (epimorphism and monomorphism).

**Conclusion**

A transformation between two vector spaces may be linear or not. A transformation between two vector spaces is linear if it obeys all the properties of a linear transformation.

The image inverse image concepts and allow the determination of the core and image.

An isomorphism \( f \) can be identified from knowledge of the core and the image off.

The relationship between the nullity feature and \( \dim(V) \), a linear transformation \( f: V \rightarrow V' \), proves to be very important, as long as they meet two of these values you can get the third, and from there, take conclusions about the characterisation of a linear transformation.

If the inverse image of a vector \( v \in V \) is different from the null image, one can conclude the core form there.

**Assessment**

Show that each of the following transformation is whether or not linear:

- \( f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), defined by \( f((x,y)) = (xy, x+y), \forall (x,y) \in \mathbb{R}^2. \)

- \( g: \mathbb{R}^3 \rightarrow \mathbb{P}^1 \), defined by \( g((a,b,c)) = (a+c)x-a+b+2c, \forall (a,b,c) \in \mathbb{R}^3. \)

Consider the linear transformation \( f: \mathbb{R}^3 \rightarrow \mathbb{P}^2 \) such that \( f((1,0,0)) = x^3 + 2x \) and \( f((0,1,0)) = x^2 - 2x \) and \( f((0,0,1)) = x^3 + x^2. \)

Determine \( f((a,b,c)) \), for all \( (a,b,c) \in \mathbb{R}. \)

Determine the core and the image of \( f. \)

Show that \( c_f + n_f = 3. \)

Consider the linear transformation \( f: \mathbb{R}^3 \rightarrow \mathbb{P}^2 \), defined by \( f((a,b,c)) = (2a+c) x^2 + (a+b)x + c, \forall (a,b,c) \in \mathbb{R}^3. \)

Show that \( f \) is an isomorphism.

Consider the linear transformation \( g: \mathbb{R}^3 \rightarrow \mathbb{P}^1 \), defined by \( f((a,b,c)) = (a+c)x-a+2b-c, \forall (a,b,c) \in \mathbb{R}^3. \)

Calculate \( f^(-1) (x-3). \)

From the result of the previous question, show that \( f \) is or not a monomorphism and...
find \( c_f \).

Determine \( f(F) \), where \( F = ((1,1,0),(-1,0,1)) \).

Determine \( f^\wedge (-1) (F') \), where \( F^\wedge = (x-1) \).

**Activity 3.2 - Matrix of a Linear Transformation: Operation with Linear Transformation**

**Introduction**

In this activity, it is intended mainly to associate a linear transformation \( f \) between real vector space of finite dimensional matrix, i.e., it will be determined matrix of a linear transformation with respect to different bases of the game space and space arrival.

Introduces the concepts of equivalent matrices and similar matrices, and shows that the relationship between each of these classes.

We also address the operations that can be made between linear transformations, and shows the relationships that exist between these operations and the matrix operations that represent the involved linear transformations.

**Activity Details**

Definition (Matrix of Linear Transformation). Let \( f:V \rightarrow V' \) be a linear transformation between vector spaces \( V \) and \( V' \) on any body \( R \), with dimensions \( n \) and \( p \), respectively. There are \( B = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \) and \( B' = \{ \vec{v}'_1, \vec{v}'_2, \ldots, \vec{v}'_p \} \) arbitrary bases of \( V \) and \( V' \), respectively. There are

\[
\begin{align*}
&f(\vec{v}_1) \equiv_{B'} (a_{11}, a_{21}, \ldots, a_{p1}) \\
&f(\vec{v}_2) \equiv_{B'} (a_{12}, a_{22}, \ldots, a_{p2}) \\
&\vdots \\
&f(\vec{v}_n) \equiv_{B'} (a_{1n}, a_{2n}, \ldots, a_{pn})
\end{align*}
\]

Then, the matrix of linear transformation \( f:V \rightarrow V' \) relative to the bases \( B \) and \( B' \) is \( A = M(f; B, B') = [a_{11} \ a_{12} \ldots a_{21} \ a_{22} \ldots : a_{p1} : a_{p2} : \ldots : a_{1n} \ a_{2n} : a_{pn}] \), that is, a matrix of dimension \( p \times n \), \( i \) is the column which comprises the vector components \( f(\vec{v}_i) \) in the base \( B' \).

Note. A linear transformation with multiple matrix, but they all have the same order where one base is chosen at the starting and another at the arrival area with given matrix or a matrix of linear transformation depends on the bases chosen in the starting space of the arrival.

Example. Let the linear transformation \( f: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) defined by

\[
f((x,y,z)) = (3x-7z, x-5y+8z),\text{ for all } (x,y,z) \in \mathbb{R}^3.
\]
Unit 3. Linear Transformation

For determining a matrix of $f$ pick up a base (arbitrary) in the starting space $\mathbb{R}^3$ and another (also arbitrary) in the arrival space $\mathbb{R}^2$.

If you choose the base $B_c = \{(1,0,0),(0,1,0),(0,0,1)\}$, standard base of $\mathbb{R}^3$, and the base $B'_c = \{(1,0),(0,1)\}$, standard base of $\mathbb{R}^2$, we have:

- $f((1,0,0)) = (3,1)$ $\equiv_{B'_c} (3,1)$
- $f((0,1,0)) = (0,-5)$ $\equiv_{B'_c} (0,-5)$
- $f((0,0,1)) = (-7,8)$ $\equiv_{B'_c} (-7,8)$

Thus, the matrix of the linear transformation $f$ with respect to these bases is

$$A = M(f; B_c, B'_c) = \begin{bmatrix} 3 & 0 & -7 \\ 1 & -5 & 8 \end{bmatrix}.$$  

On the other hand, if one chooses the base $B = \{(1,-1,-1),(0,1,2),(0,0,1)\}$ of $\mathbb{R}^3$, and the base $B'^\prime = \{(1,-1),(-1,2)\}$ of $\mathbb{R}^2$, we have:

- $f((1,-1,-1)) = (10,-2)$
- $f((0,1,2)) = (-14,11)$
- $f((0,0,1)) = (-7,8)$

The components of these images in the base $B'$ is determined as follows:

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 10 & -14 \\ -2 & -7 & 8 \\ 11 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 10 & -14 & -7 \\ 8 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 18 & -17 & -6 & 8 \\ -3 & 1 \end{bmatrix}$$

Then:

- $f((1,-1,-1)) = (10,-2) \equiv_{B'} (18,8)$
- $f((0,1,2)) = (-14,11) \equiv_{B'} (-17,-3)$
- $f((0,0,1)) = (-7,8) \equiv_{B'} (-6,1)$

Therefore, the matrix of the linear transformation $f$ with respect to these bases is:

$$A = M(f; B, B'^\prime) = \begin{bmatrix} 18 & -17 & -6 & 8 \\ -3 & 1 \end{bmatrix}.$$  

Theorem. Let $f: V \rightarrow V'$ be a linear transformation between vector spaces $V$ and $V'$ on the body $\mathbb{R}$, with dimensions $n$ and $p$, respectively. There are two arbitrary bases of $V$ and $V'$, given a vector $x \in V$, let $x \equiv_B (x_1, x_2, \cdots, x_n)$ (vector components in the base $B$). Consider the column matrix $X = [x_1, x_2, \cdots, x_n]^t$, $x'$ of the vector components base $B$. Then $X'^\prime = AX$ is the column matrix of components of $f(x')$ on the base $B'$.

Example. $f$ is a linear transform of the previous example. The matrix of $f$ with respect to the bases $B = \{(1,-1,-1),(0,1,2),(0,0,1)\}$ of $\mathbb{R}^3$ and $B'^\prime = \{(1,-1),(-1,2)\}$ of $\mathbb{R}^2$ is, as we know:

$$A = M(f; B, B'^\prime) = \begin{bmatrix} 18 & -17 & -6 & 8 \\ -3 & 1 \end{bmatrix}.$$  

Let $x = (x,y,z)$ be an arbitrary vector $\mathbb{R}^3$. Determine $x'$ the vector components in the base
\[ B = \{(1, -1, -1), (0, 1, 2), (0, 0, 1)\}, \] as follow:
\[
\begin{bmatrix}
1 & 0 & 0 & -1 & x & y & z \\
1 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & x & y & x & z \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & x & y & -x & 2y & +z
\end{bmatrix}
\]

Then, \( x' = (x, y, z) \equiv_B X = (x, x+y, -x+2y+z) \). Calculate
\[
X^\prime = AX = [7x-5y-6z 4x-5y+z ].
\]
That is, \( f((x, y, z)) \equiv_{B'} (7x-5y-6z, 4x-4y+z) \). Then
\[
f((x, y, z)) = (7x-5y-6z)(1, -1) + (4x-4y+z)(-1, 2) = (3x-7z, x-5y+8z)
\]

Note. The previous example show that a linear transformation \( f:V \rightarrow V^{\prime} \) is well defined when it meets a matrix \( f \) with respect to arbitrary bases of \( V \) and \( V^{\prime} \).

Every matrix \( A \) of dimension \( m \times n \) is a given linear transformation \( f:V \rightarrow V^{\prime} \), where \( \dim(V) = n \) e \( \dim(V^{\prime}) = m \). The linear transformation is completely defined when one knows the bases \( B \) and \( B^{\prime} \), such that \( A = M(f; B, B^{\prime}) \).

Example. Let the real matrix be
\[
A = \begin{bmatrix}
-1 & 0 & -2 & 2 & 1 & 5
\end{bmatrix}
\]
The matrix \( A \) is a linear transformation \( f:V \rightarrow V_{\prime} \), where \( \dim(V) = 3 \) e \( \dim(V_{\prime}) = 2 \), for example, \( f:R^3 \rightarrow P^1 \).

Select an arbitrary bases \( V \), and another \( V' \). In this case, either,
\[
B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\},
\]
a canonical base of \( R^3 \), and
\[
B^{\prime} = \{x, 1\},
\]
a base of \( P^1 \).

Then, for any \( (a, b, c) \in R^3 \), \( (a, b, c) \equiv_B (a, b, c) \). It has:
\[
A[a b c ] = [-a-2c 2a+b+5c ].
\]
Soon,
\[
f((a, b, c)) = (-a-2c)x+2a+b+5c.
\]

Theorem. Let \( f:V \rightarrow V' \) be a linear transformation between the real vector spaces \( V \) and \( V' \). Let \( B \) and \( B' \) be arbitrary bases of \( V \) and \( V' \) respectively. Then, \( f \) is an isomorphism if and only if, \( A = M(f; B, B') \) is invertible (or regular).

Another way to define a matrix base change

Definition. Let \( V \) be a \( n \in N \). Let the linear transformation identity
\[
i:V \rightarrow V.
\]
Consider \( V \) a dual bases \( B \) and \( B' \), starting at the first space and the second space of chagada.
Then, the matrix

\[ M(i,B,B') = M(B' \rightarrow B) \quad e \quad M(i,B',B) = M(B \rightarrow B'). \]

Operations with linear transformations

Definition. Let V and V' be a real vector spaces, f:V\rightarrow V' e g:V\rightarrow V' linear transformations.

1. The sum of f with g is a linear transformation f+g:V\rightarrow V' defined by (f+g)(x⃗)=f(x⃗)+g(x⃗), ∀ x⃗∈V.

2. The result of the application f by a scalar λ∈R is a linear transformation λf:V\rightarrow V' defined by (λf)(x⃗)=λf(x⃗), ∀ x⃗∈V.

Theorem. In the above definition of terms, if B and B' are bases of V and V', respectively, and A=M(f;B,B') and C=M(g;B,B'), then

A+C=M(f+g;B,B') \quad and \quad λA=M(λf;B,B').

Definition. Let V, V' and V'' be real vector spaces, f:V\rightarrow V' and g:V'\rightarrow V'' be a linear transformation. The g\circ f composition is a linear transformation g\circ f:V\rightarrow V'' defined by (g\circ f)(x⃗)=g(f(x⃗)), ∀ x⃗∈V.

Theorem. In the above definition of terms, if B, B' and B'' are bases of V, V' and V'', respectively, and A=M(f;B,B') e C=M(g;B',B''), then

CA=M(g\circ f;B,B'').

Theorem. Let f:V\rightarrow V' be an isomorphism, between the real vector spaces V and V'. Then, applying the inverse f^(-1):V'\rightarrow V is an isomorphism. Moreover, if A=M(f;B,B'), then

A^(-1)=M(f;B',B).

Example. The following linear transformation are:

f:R^3\rightarrow P^2, defined by

\[ f((a,b,c))=(a+c)x^2+(b+c)x+a-b+c, \forall (a,b,c)∈R^3; \]

\[ g:R^3\rightarrow P^2, defined by the matrix \]

\[ A=M(g;B,B')=[1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 ]; \]

where B={⟨1,1,1⟩,⟨0,1,2⟩,⟨-1,0,2⟩} is a base of R^3 and B^'={x^2,-x,x+1} is a base of P^2;

h:P^2\rightarrow R, defined by

\[ h(ax^2+bx+c)=a+b-c, \forall ax^2+bx+c∈P^2. \]
Determine the matrix $A^\prime = M(f; B, B')$ and the matrix $C = M(h; B^\prime, B^\prime')$, where $B^\prime'=\{1\}$ is a base of $R$.

Define the linear transformation $f + g : R^3 \rightarrow P^2$.

Determine the matrix $D = M(f + g, B, B')$ and show that $D = A + A'$.

Show that $f$ is an isomorphism.

Define the isomorphism $f^\wedge(-1): P^2 \rightarrow R^3$.

Determine the application of the matrix $-3(f + g): R^3 \rightarrow P^2$.

Determine the matrix $M(h \circ g; B, B^\prime)$; and define the application $h \circ g : R^3 \rightarrow R$.

**Determination of the matrix $A^\prime = M(f; B, B')$.**

\[
\begin{align*}
&f((1,1,1))=2x^2+2x+1; 
f((0,1,2))=2x^2+3x+1; 
f((-1,0,2))=x^2+2x+1 \\
&\text{Determine the component of these vectors in the base } B^\prime=\{x^2,-x,x+1\}: \\
&\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
2 & 2 & 1 & 2 \\
3 & 2 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} \sim \\
&\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
2 & 2 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} \sim \\
&\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
2 & 1 & 1 & 2 \\
1 & 1 & 1 & 1
\end{bmatrix} \\
\text{Thus, } f((1,1,1)) \equiv_{B'} (2,-1,1); 
f((0,1,2)) \equiv_{B'} (2,-2,1); 
f((-1,0,2)) \equiv_{B'} (1,-1,1). \quad \text{Soon:} \\
&A^\prime= M(f; B, B')=[2 2 1 -1 -2 -1 1 1 1].
\end{align*}
\]

**Determination of the matrix $C=M(h; B^\prime, B^\prime')$.**

\[
\begin{align*}
&h(x^2)=1 \equiv_{B''} (1); 
h(-x)=-1 \equiv_{B''} (-1); 
h(x+1)=0 \equiv_{B''} (0). \quad \text{Logo} \\
&C= M(h; B^\prime, B^\prime')=[1 -1 0].
\end{align*}
\]

**Definition of linear transformation $f+g : R^3 \rightarrow P^2$.**

First, define the linear transformation $g$ through their expression of the image. To do this determine the vector components $(a,b,c)$ (arbitrary vector of $R^3$) at the base $B$.

\[
\begin{align*}
&[1 0 1 0 1 2 2 a b c] \sim [1 0 -1 0 1 0 0 1 2 a-b-a] \sim [1 0 1 0 0 1 0 a-b-a a-2b+c] \sim \\
&[1 0 0 0 1 0 0 0 1 2a-2b+c -2a+3b-c a-2b+c]
\end{align*}
\]

Therefore, $(a,b,c) \equiv_{B} (2a-2b+c,-2a+3b-c,a-2b+c)$. It is now estimated,

\[
\begin{align*}
&A[2a-2b+c -2a+3b-c a-2b+c]=[1 0 0 0 1 0 1 0 0 1 2a-2b+c -2a+3b-c a-2b+c]=2a-2b+c \\
&\quad \quad -2a+3b-c a-2b+c 4a-5b+2c \\
\begin{align*}
\text{Then, } g((a,b,c)) &\equiv_{B'} (2a-2b+c,-2a+3b-c,4a-5b+2c). \quad \text{Now:} \\
g((a,b,c)) &= (2a-2b+c)x^2+(2a-3b+c)x+(4a-5b+2c)(x+1) \\
&= (2a-2b+c)x^2+(6a-8b+3c)x+4a-5b+2c \\
f((a,b,c))+g((a,b,c)) &= (3a-2b+2c)x^2+(6a-7b+4c)x+5a-6b+3c
\end{align*}
\]
Then the linear transformation \( f+g: \mathbb{R}^3 \to \mathbb{P}^2 \) is defined by

\[
(f+g)(a,b,c) = (3a-2b+2c)x^2+(6a-7b+4c)x+5a-6b+3c,
\]

\( \forall (a,b,c) \in \mathbb{R}^3 \).

Determination of the matrix \( D=M(f+g,B,B') \).

\[
(f+g)(1,1,1) = 3x^2+3x+2
\]

\[
(f+g)(0,1,2) = 2x^2+x
\]

\[
(f+g)(-1,0,2) = x^2+2x+1
\]

Determine the components of these vectors in the base \( B' \).

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
3 & 2 & 1 & -1 & -1 & -1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
3 & 2 & 1 & -1 & -1 & -1 & 2 & 0 & 1
\end{bmatrix}
\]

\[
(f+g)(1,1,1) = 3x^2+2x+4 \equiv _{B'} (3,-1,2)
\]

\[
(f+g)(0,1,2) = 2x^2-x+4 \equiv _{B'} (2,-1,0)
\]

\[
(f+g)(-1,0,2) = x^2+1 \equiv _{B'} (1,-1,1)
\]

\[
D=M(f+g,B,B') = [3 2 1 -1 -1 -1 2 0 1].
\]

On the other hand,

\[
A+A' = [1 0 0 0 1 0 1 0 0 ]+[2 2 1 -1 1 1 1 1 1 ] = [3 2 1 -1 -1 -1 2 0 1]=D.
\]

f is an isomorphism? As already know, a matrix of f, \( A' = M(f;B,B') \), it can be said that f is isomorphism if \( A' \) is invertible. \( A' \) is invertible if and only if, \( |A'| \neq 0 \).

\[|A'| = -3-4-(-2-2) = -7+4 = -3 \]

Now \( A' \) is invertible and hence f é is an isomorphism.

Definition of isomorphism \( f^\wedge(-1):\mathbb{P}^2 \to \mathbb{R}^3 \).

\[f^\wedge(-1)(a,b,c) = [1 0 0 -1 0 1 -1 1 1 ] \to [1 0 0 -1 0 1 -1 1 1 ] = [3 2 1 -1 -1 -1 2 0 1].
\]

Determine the components of the arbitrary vector \( ax^2+bx+c \) de \( \mathbb{P}^2 \), na base \( B' = \{x^2,-x,x+1\} \).

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\
a & b & c
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 1 0 0 1 a & b & c
\end{bmatrix}
\]

\[
ax^2+bx+c \equiv _{B'} (a,-b+c,c)
\]

Determine the component vector \( f^\wedge(-1) (ax^2+bx+c) \).

\[
[f^\wedge(-1) (a-b+c c)] = [1 1 0 0 -1 1 0 2 2 ] [a -b+c c ] = [a-b+c b-2c-a+2c].
\]

f\(^\wedge(-1)\) \( (ax^2+bx+c) \equiv _{B'} \) \( (a-b+c,-b+a-2c) \). Então:

\[
f^\wedge(-1) (ax^2+bx+c) = (a-b+c)(1,1)+(b-2c)(0,1,2)(-a+2c)(1,0,2)
\]

\[
= (2a-b-c,a-c,-a+b+c)
\]

Determining the linear transformation matrix \(-3(f+g):\mathbb{R}^3 \to \mathbb{P}^2\).

\[-3D=M(-3(f+g),B,B') = [-9 -6 3 3 3 -6 0 -3].
\]
Determining the matrix $M(h \circ g; B, B'')$; and definition of the transformation $h \circ g : \mathbb{R}^3 \rightarrow \mathbb{R}$.

As
$$A = M(g; B, B') = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = M(h; B', B'') = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix},$$

Then
$$M(h \circ g; B, B'') = CA = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}.$$

As $(a, b, c) \equiv_B (2a-2b+c, -a+3b-c, a-2b+c)$ and
$$M(h \circ g; B'') \begin{bmatrix} 2a-2b+c \\ -a+3b-c \\ a-2b+c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2a-2b+c \\ -a+3b-c \\ a-2b+c \end{bmatrix} = 4a-5b+2c.$$

As $B'' = \{1\}$, then
$$(h \circ g)(a, b, c) = 4a-5b+2c, \quad \forall (a, b, c) \in \mathbb{R}^3.$$

Relationship between two matrices of a linear transformation

Let a linear transformation $f : V \rightarrow V'$, between the real vector spaces $V$ and $V'$, of dimensions $n$ and $m$, respectively. Let $A = M(f; B_1, [B']_1)$ and $C = M(f; B_2, [B']_2)$, where $B_1$ and $B_2$ are arbitrary bases of $V$ and $[B']_1$ and $[B']_2$ are arbitrary bases of $V$. Let $P = M(i; B_2, B_1)$ and $Q = M(i; [B']_1, [B']_2)$.

Then
$$C = QAP \quad \Leftrightarrow \quad A = Q^(-1) CP^(-1).$$

Definition (Equivalent Matrices). Let $A$ and $C$ be the real type of matrices $m \times n$. It is said that $A$ is equivalent to $C$, and represented by $A \sim C$, if there are invertible matrices $P$, of order $n$, and $Q$, of order $m$, such that $C = QAP$.

Theorem. Let $A$ and $C$ be matrices of dimensions $m \times n$. Then $A \sim C$ if and only if there exist a linear transformation $f : V \rightarrow V'$, between real vector spaces $V$ and $V'$, of dimensions $n$ and $m$, respectively, and are of bases, $B_1$ and $B_2$ of $V$, and $[B']_1$ and $[B']_2$ of $V'$, such that $A = M(f; B_1, [B']_1)$ and $C = M(f; B_2, [B']_2)$.

Note. According to the previous theorem, if the two matrices are equivalent, so it represents a particular linear transformation with respect to certain bases of the starting space and the arrival space. Also, a given arbitrary linear transformation, are equivalent to any two of their matrices, on certain bases of the starting space and the arrival space.

Example. Find an equivalent matrix to
$$A = \begin{bmatrix} -1 & 1 & 1 & 0 & 2 \end{bmatrix} \in \mathbb{R}^{2 \times 3}.$$

Now, $A$ represents a linear transformation $f : V \rightarrow V'$, where $\dim(V) = 3$ and $\dim(V') = 2$. The, let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, for example. One can consider that $A = M(f; B, B')$, where $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $B'' = \{(1,0),(0,1)\}$ are standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

For all $(x, y, z) \in \mathbb{R}^3$, $(x, y, z) \equiv_B (x, y, z)$ and
$$A[x y z] = [-x+y+z x+2z],$$

It has
$$f((x, y, z)) = (-x+y+z)(1,0)+(x+2z)(0,1) = (-x+y+z, x+2z).$$
Choose a base $B_1$ of $\mathbb{R}^3$ and the base $[B' ]_1$ of $\mathbb{R}^2$, such that $B_1 \neq B$ or $[B' ]_1 \neq B'$. For example, $B_1=\{(1,-2,0),(0,1,-1),(1,-2,1)\}$ and $[B' ]_1=\{(1,1),(-1,1)\}$.

Let

$$B=M(f;B_1,[B' ]_1)=[-1 -1 1/2 2 -1 5/2 ].$$

The matrices $A$ and $B$ are equivalent. Note that $B=QAP$, where

$$Q=M(i,B^',[B' ]_1)=[1/2 1/2 -1/2 1/2 ] \quad \text{and} \quad P=M(i,B_1,B)=[1 0 1 -2 1 -2 0 -1 1 ],$$

are invertible matrices, since $|Q|=1/2$ and $|P|=1$.

Let $V$ be a vector space of dimension $n$, and $f:V \rightarrow V$, be an endomorphism of $V$. Let $B_1$ and $B_2$ arbitrary bases of $V$. Let $A=M(f;B_1,B_1)$, $C=M(f;B_2,B_2)$, $P=M(f;B_2,B_1)$ and $Q=P^{-1}=M(i,B_1,B_2)$. Then

$$C=QAP \iff C=P^{-1}AP.$$

Definition (Similar Matrices). Let $A$ and $C$ be real square matrices of order $n$. It is said that $A$ is similar to $B$, and is represented by $A \approx C$, if there exists an invertible matrix $P$, such that $C=P^{-1}AP$.

Theorem. Let $A$ and $C$ be real matrices of order $n$. Then $A \approx C$ if and only if there exist an endomorphism $f:V \rightarrow V$, where $V$ is a real vector space of dimension $n$, and there are of bases $B_1$ and $B_2$ of $V$, such that $A=M(f;B_1,B_1)$ and $C=M(f;B_2,B_2)$.

Note. According to the previous theorem, if the two matrices are similar, so they represent a particular endomorphism of of a real vector space $V$, with respect to two bases $V$. Also, since an endomorphism of a real vector space $V$, are any two similar matrices with respect to two bases of $V$.

Example. Giving example similar square matrices $A$ and $B$, of order $3$.

However, the matrices $A$ and $B$ represent endomorphism $f:V \rightarrow V$, where $\dim(V)=3$ Let, for example, $f:P^2 \rightarrow P^2$ (you can also choose, $f:R^3 \rightarrow R^3$).

The chosen square matrix of order $3$; for example

$$A=[1 -1 0 1 0 1 1 2 3 ].$$

One can consider that $A=M(f;B,B)$, where $B=\{x^2,x,1\}$ is the standard base of $P^2$.

For all $ax^2+bx+c \in P^2$, $ax^2+bx+c \equiv _B (a,b,c)$ and

$$A[a b c ]=[a-b a+c a+2b+3c ],$$

Now

$$f(ax^2+bx+c)=(a-b)x^2+(a+c)x+a+2b+3c.$$  

Choose any basis $B_1$ and $P^2$, such that $B_1 \neq B$; for example $B_1=\{x^2-1,x+1,1\}$.  


Let
\[ B = M(f; B_1, B_1) = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 1 & -1 & 3 & 2 \end{bmatrix}. \]

The matrices \( A \) and \( B \) are similar. Note that \( B = P^(-1) AP \), where
\[ P = M(i; B_1, B) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad P^(-1) = M(i; B, B_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1 \end{bmatrix}. \]

**Conclusion**

Any linear transformation \( f: V \to V' \), between vector space of finite dimensions \( V \) and \( V' \), can be associated with different matrices of the same order (as is the chosen bases in \( V \) and \( V' \)). If \( \dim(V) = n \) and \( \dim(V') = m \), then any matrix that represents \( f \) has dimension \( m \times n \).

A linear transformation is completely set when a base is known of starting space, a basis of the arrival space and the matrix that is, in relation to these bases.

Every matrix of dimension \( m \times n \), may represent a linear transformation \( f: V \to V' \), where \( \dim(V) = n \) and \( \dim(V') = m \).

Every isomorphism is represented by invertible matrices, and all invertible matrix represent an isomorphism. Consequently, every change of basis matrix is invertible, and the entire matrix is an invertible basis change matrix.

The sum of two linear transformations, involves the addition of corresponding matrices in relation to appropriate bases. The composition of transformations requires multiplication of their respective plates in reverse order in relation to appropriate bases. The multiplication of a linear transformation by a scalar, is increasing the number of this scale by their matrices respectively.

Given a rectangular matrix, it is always possible to obtain matrices equivalent to it. Given a square matrix, it is always possible to obtain matrices similar to it.

**Assessment**

Let a linear transformation \( f: \mathbb{R}^3 \to \mathbb{P}^1 \), be such that
\[ A = M(f; B, B') = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & -2 \end{bmatrix}, \]
where \( B = \{(1,-1,1),(0,2,-1),(1,-1,2)\} \) e \( B' = \{x+1,x-1\} \).

Show that for all \( a, b, c \in \mathbb{R} \), \( (a,b,c) \equiv_B (3/2 a - 1/2 b - c, 1/2 a + 1/2 b, -1/2 a + 1/2 b + c) \).

Show that for all \( a, b, c \in \mathbb{R} \), \( f((a,b,c)) = (-a+b)x - a + b + 2c \).

Determine the matrix \( C = M(f; B_1, (B')_1) \), where \( B_1 = \{(1,0,0),(0,1,0),(0,0,1)\} \) and \( (B')_1 = \{x,-x+1\} \).

Determine the invertible matrices \( Q \), of order 2, and \( P \), of order 3, such that \( C = QAP \).

Determine a similar matrix to the matrix \( A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ 1 & 3 \end{bmatrix} \).

Let the linear transformation \( g: \mathbb{P}^2 \to \mathbb{R}^3 \) such that \( A = M(g; B, B') = \begin{bmatrix} -1 & 1 & 1 & 0 & -1 & -2 & 1 \end{bmatrix} \),
where \( B = \{ x^2, x-1, -x^2+1 \} \) and \( B' = \{(1,0,1),(-1,1,-1),(1,-1,2)\} \).

- Justify that \( g \) is an isomorphism.
- Define the isomorphism \( g^\star : \mathbb{R}^3 \to \mathbb{P}^2 \).
- Let \( h : \mathbb{R}^3 \to \mathbb{P}^1 \), such that \( h((a,b,c)) = (a+b-c)x-a+2c \), for all \( a, b, c \in \mathbb{R} \). Define \( h \circ g \).
- Determine the matrix \( D = M(h \circ g; B', B'' \}) \), where \( B'' = \{ x, 1 \} \).

Activity 3.3 - Adjoint of a Linear transformation - adjoint endomorphism, orthogonal endomorphism

**Introduction**

In this activity we study the adjoint of a linear transformation, especially adjoint and self-adjoint endomorphism.

The emphasis is the study of orthogonal endomorphism in plane and in Euclidean space: reflection and rotation.

**Activity Details**

**Definition (Adjoint of a linear transformation).** Let \( f : V \to V' \) be a linear transformation between the real vector spaces \( V \) and \( V' \), with inner product. A linear transformation \( f^\star : V' \to V \) is said to be adjunct of \( f \) if \( \forall v \in V, \forall v' \in V' \), then we have
\[
f(v) \cdot v' = v \cdot f^\star (v').
\]

**Theorem.** The adjoint of a linear transformation exist when it is unique.

**Theorem.** Let \( f : V \to V' \) be a linear transformation of real vector space of finite dimension of \( V \) into \( V' \), with inner product, i.e., \( V \) and \( V' \) are Euclidean spaces. Then \( f \) has a unique adjoint.

**Theorem.** Let \( V \) and \( V' \) be an Euclidean space of dimension \( n \) and \( m \), with bases \( B = \{ v_1, v_2, \ldots, v_n \} \) and \( B' = \{ v'_1, v'_2, \ldots, v'_m \} \), respectively. Let \( G \) and \( G' \) be metric matrices of the inner product in relation to the base \( V \) and \( V' \), respectively. Let \( f : V \to V' \) be a linear transformation and \( A = [a_{ij}] = M(f; B, B') \) (matrix of dimension \( m \times n \)). then:
\[
A' = M(f^\star; B'B) = [G']^\star (-1) A^\star G, \text{ where } f^\star : V' \to V \text{ is the adjoint } f : V \to V'.
\]

If the bases \( B \) and \( B' \) are orthonormal, then \( G = I_n \) and \( G' = I_m \), logo \( A' = A^\star \).

**Note.** Given the Euclidean spaces \( V \) and \( V' \), and a linear application \( f : V \to V' \), you can apply those rules stipulated in the previous example to determine that its adjoint is \( f^\star : V' \to V \).

**Exemple.** Let the real vector spaces be \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \), containing the canonical inner product. Let the linear transformation be \( f : \mathbb{R}^3 \to \mathbb{R}^2 \), defined by
\[
f((x,y,z)) = (x-y+z, y+2z).
\]
Determine the adjoint of \( f \).

In relation with the canonical domestic product, 
\[ B=\{(1,0,0),(0,1,0),(0,0,1)\} \] and \( B^\prime=\{(1,0),(0,1)\} \),
are orthonormal bases of \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \), respectively. Then
\[ A=M(f;B,B')=[1 -1 1 0 1 2]. \]

Now,
\[ A'=M(f^*;B^',B)=A^t=[1 0 -1 1 2]. \]

For all, \( a,b\in\mathbb{R} \), \( (a,b)\equiv_{B'}(a,b) \).
\[ A'[a b]=[a -a+b a+2b]. \] Therefore \( f^*:\mathbb{R}^2\rightarrow\mathbb{R}^3 \), is defined by \( f^* ((a,b))=(a,-a+b,a+2b) \).

In fact, for all \( (x,y,z)\in\mathbb{R}^3 \), \( (a,b)\in\mathbb{R}^2 \), we have:
\[ f((x,y,z))\cdot(a,b) = (x-y+z,y+2z)\cdot(a,b) = (x-y+z)a+(y+2z)b = ax+(-a+b)y+(a+2b)z \]
\[ f^* ((a,b))\cdot(x,y,z) = (a,-a+b,a+2b)\cdot(x,y,z) = ax+(-a+b)y+(a+2b)z, \]
that is,
\[ f((x,y,z))\cdot(a,b)=f^* ((a,b))\cdot(x,y,z). \]

Definition (Adjoint Endomorphism). Let \( f:V\rightarrow V \), be an endomorphism of the Euclidean space \( V \), of dimension \( n \). It is said that the endomorphism \( f^*:V\rightarrow V \) is an adjoint endomorphism of \( f \) if for any vectors \( v,w\in V \), we have
\[ f(v)\cdot w=v\cdot f^*(w). \]

Note (Very important!!!). The previous theorem, if \( f:V\rightarrow V \), is an endomorphism of the Euclidean space \( V \), \( B \) is an arbitrary base of \( V \), \( G \) is a metric matrix of the inner product \( V \), in relation to the base \( B \), and \( A=M(f;B,B) \), then:
\[ A'=M(f^*,B,B)=G^(-1) A^t G, \] if \( B \) is an orthonormal base of \( V \), then \( G=G^(-1)=I \), then \( A'=A^t \).

Exemple. Let \( V \) be an Euclidean space of dimension 2. Let \( G=[1 0 0 -1] \) be a métric matrix of the inner product \( V \), in relation to a particular base \( B=\{v^\prime,1,v^\prime,2\} \). Let \( f:V\rightarrow V \), be an endomorphism of \( V \) and \( A=M(f;B,B)=[1 -1 3 2] \). Define an endomorphism \( f^* \) (adjoint of \( f \)).

The previous note, \( C=M(f^*,B,B)=G^(-1) A^t G \). Since \( G^(-1)=G \), then
\[ A'=[GA]^t G=[1 1 -3 2]. \]
Let \( \vec{v} \) be an arbitrary vector of \( V \); since \( B=\{\vec{v}_1,\vec{v}_2\} \) is a base of \( V \), then there exist a scalar \( \lambda,\beta \in \mathbb{R} \), such that \( \vec{v} = \lambda \vec{v}_1 + \beta \vec{v}_2 \). We have

\[
A'[\lambda,\beta] = \begin{bmatrix} \lambda + \beta & -3\lambda + 2\beta \end{bmatrix}.
\]

then, \( f^*:V \rightarrow V \) is defined by

\[
f^*(\vec{v}) = (\lambda + \beta)\vec{v}_1 + (-3\lambda + 2\beta)\vec{v}_2.
\]

Definition (Self-adjoint Endomorphism). An endomorphism that is attached to itself is called self-adjoint endomorphism.

let \( f:V \rightarrow V \), be an self-adjoint endomorphism of the euclidean space of dimension \( n \). Let \( B=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\} \) be an arbitrary base of \( V \) and \( A=M(f;B,B) \). Since an adjoint of \( f \) is \( f \) itself, then

\[
A^*=M(f^*;B,B) = A,
\]

and we have:

\[
A = G^{-1} A^t G,
\]

where \( G \) is a metric matrix of the inner product \( V \), considered in relation to the base.

If \( B=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\} \) is an orthonormal base, we have \( A = A^t \), that is, a self-adjoint matrix endomorphism, in relation to an orthonormal, is symmetrical.

Theorem. Let \( f, g \) be a linear transformation between appropriate Euclidean space, then:

\[
\begin{align*}
(f+g)^* &= f^* + g^*; \\
(\lambda f)^* &= \lambda f^*; \text{ for all } \lambda \in \mathbb{R}; \\
(fg)^* &= g^* f^*;
\end{align*}
\]

\( i^* = i \), where \( i \) is transformation identity.

Definition (Orthogonal Endomorphism). Let \( f:V \rightarrow V \), be an endomorphism of a Euclidean space \( V \). It is said that \( f \) is an orthogonal endomorphism if \( \forall \vec{v}, \vec{w} \in V \), we have

\[
f(\vec{v}) \cdot f(\vec{w}) = \vec{v} \cdot \vec{w}.
\]

Theorem. Let \( f:V \rightarrow V \), be an endomorphism of an Euclidean space \( V \). Then:

If for any \( \vec{v} \in V \) \( ||f(\vec{v})|| = ||\vec{v}|| \), then \( f \) is an orthogonal endomorphism;

\( f \) is an orthogonal endomorphism if and only if the matrix \( f \), with respect to any orthonormal basis \( V \), is orthogonal.

Examples of orthogonal endomorphisms in Euclidean plane \( \mathbb{R}^2 \).

Let the vector space be \( \mathbb{R}^2 \), provided that it is an Euclidean inner product.

Rotation by an angle \( \theta \), around the origin \( O \), in an anticlockwise direction. Let an endomorphism \( R_(\theta,O):\mathbb{R}^2 \rightarrow \mathbb{R}^2 \), which lead to all vector \( \vec{v} = (v_x,v_y) \in \mathbb{R}^2 \) in its rotation by an angle \( \theta \), around \( O \), in an anticlockwise direction, that is,

\[
R_(\theta,O) ((v_x,v_y)) = (v_x \cos(\theta) - v_y \sen(\theta), v_x \sen(\theta) + v_y \cos(\theta)).
\]

This endomorphism is orthogonal, because \( ||R_(\theta,O) ((v_x,v_y))|| = ||(v_x,v_y)|| \).
Note that the matrix
\[ A = M(R_x;B,B) = [\cos(\theta) \cdot \sin(\theta) \cdot \sin(\theta) \cdot \cos(\theta) ] , \]
where \( B = \{i=(1,0), j=(0,1)\} \).

Reflections in relation to the coordinate axes. Let the endomorphisms \( R_x: \mathbb{R}^2 \to \mathbb{R}^2 \), \( R_y: \mathbb{R}^2 \to \mathbb{R}^2 \) leading all vector \( v = (v_x, v_y) \in \mathbb{R}^2 \) be the reflection about the x and y, respectively, that is,
\[ R_x ((v_x,v_y)) = (v_x, -v_y) \quad \text{and} \quad R_y ((v_x,v_y)) = (-v_x, v_y). \]
These endomorphisms are orthogonal because \( \|R_x ((v_x,v_y))\| = \|(v_x, -v_y)\| \) and \( \|R_y ((v_x,v_y))\| = \|(-v_x, v_y)\| \).

Note that matrices
\[ A = M(R_x;B,B) = [1 \begin{array}{c} 0 \end{array} ] \quad \text{and} \quad A = M(R_y;B,B) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} , \]
where \( B = \{i=(1,0), j=(0,1)\} \).
Reflection in relation to the line $y=x$. Let the linear transformation $R_{(y=x)}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which leads to all vector $\mathbf{v}=(v_x,v_y) \in \mathbb{R}^2$ be the reflection about $y=x$, that is, 

$$R_{(y=x)}((v_x,v_y))=(v_y,v_x).$$

These endomorphism is orthogonal, because $\|R_{(y=x)}((v_x,v_y))\| = \|(v_y,v_x)\|$.

Note that the matrix

$$A=M(R_{(y=x)};B,B)=[0 1 1 0],$$

where $B=[(1,0),(0,1)]$.

Examples of orthogonal endomorphisms in Euclidean space $\mathbb{R}^3$.

Let the vector space be $\mathbb{R}^3$, provided with Euclidean inner product.

Rotation about the $x$ axis, let the endomorphism $R_{(\theta,x)}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that takes all vectors $\mathbf{v}=(v_x,v_y,v_z) \in \mathbb{R}^3$ 

Rotation by an angle $\theta$, around the $y$ axis, in an anticlock direction. Let the endomorphism $R_{(\theta,y)}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which takes all vectors $\mathbf{v}=(v_x,v_y,v_z) \in \mathbb{R}^3$ be in its rotation by an angle $\theta$, around the $y$ axis, in anticlockwise direction, that is, 

$$R_{(\theta,y)}((v_x,v_y,v_z))=(v_x \cos(\theta)+v_z \sin(\theta),v_y,-v_x \sin(\theta)+v_z \cos(\theta)).$$

This endomorphism is orthogonal, because $\|R_{(\theta,y)}((v_x,v_y,v_z))\| = \|(v_x,v_y,v_z)\|$.

Note that the matrix

$$A=M(R_{(\theta,y)};B,B)=[\cos(\theta) 0 \sin(\theta) 0 1 0 \sin(\theta) 0 \cos(\theta)],$$

where $B=[(1,0,0),(0,1,0),(0,0,1)]$.

Rotation by an angle $\theta$, around the $x$ axis, in an anticlockwise direction. Let the endomorphism $R_{(\theta,x)}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which takes all vectors $\mathbf{v}=(v_x,v_y,v_z) \in \mathbb{R}^3$ be in its rotation by an angle $\theta$, around the $x$ axis, in the anticlockwise direction, that is, 

$$R_{(\theta,x)}((v_x,v_y,v_z))=(v_x,v_y \cos(\theta)-v_z \sin(\theta),v_y \sin(\theta)+v_z \cos(\theta)).$$

This endomorphism is orthogonal, because $\|R_{(\theta,x)}((v_x,v_y,v_z))\| = \|(v_x,v_y,v_z)\|$.
Note that the matrix
\[ A = M(R(\theta,x);B,B) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]
where \( B = \{i = (1,0,0), j = (0,1,0), k = (0,0,1)\}. \)

Rotation by an angle \( \theta \), around the \( z \) axis, in an anticlockwise direction. Let the endomorphism \( R(\theta,z):R^3 \to R^3 \) which takes all vectors \( \vec{v} = (v_x,v_y,v_z) \in R^3 \) be its rotation by an angle \( \theta \), around the \( z \) axis, in the anticlockwise direction, that is,
\[
R_\theta(\vec{v}) = (v_x \cos(\theta) - v_y \sin(\theta), v_x \sin(\theta) + v_y \cos(\theta), v_z).
\]
This endomorphism is orthogonal, because \( ||R_\theta(\vec{v})|| = ||\vec{v}|| \).

Note that the matrix
\[ A = M(R(\theta,z),B,B) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]
where \( B = \{i = (1,0,0), j = (0,1,0), k = (0,0,1)\}. \)

Reflection to the plane \( z=0 \). Let the endomorphism \( R(z=0):R^3 \to R^3 \), which takes all vectors \( \vec{v} = (v_x,v_y,v_z) \in R^3 \) be their reflection to the plane, that is,
\[
R_{z=0}(\vec{v}) = (v_x, v_y, -v_z).
\]
This endomorphism is orthogonal, because \( ||R_{z=0}(\vec{v})|| = ||\vec{v}|| \).

Note that the matrix
\[ A = M(R(z=0),B,B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \]
where \( B = \{i = (1,0,0), j = (0,1,0), k = (0,0,1)\}. \)

Reflection to the plane \( y=0 \). Let the endomorphism \( R(y=0):R^3 \to R^3 \), which takes all vectors \( \vec{v} = (v_x,v_y,v_z) \in R^3 \) be their reflection to the plane \( y=0 \), that is,
\[
R_{y=0}(\vec{v}) = (v_x, -v_y, v_z).
\]
The endomorphism is orthogonal, because \( ||R_{y=0}(\vec{v})|| = ||\vec{v}|| \).

Note that the matrix
\[ A = M(R(y=0),B,B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]
where \( B = \{i = (1,0,0), j = (0,1,0), k = (0,0,1)\}. \)

Reflection to the plane \( x=0 \). Let the endomorphism \( R(x=0):R^3 \to R^3 \), which takes all vectors \( \vec{v} = (v_x,v_y,v_z) \in R^3 \) be their reflection to the plane \( x=0 \), that is,
\[
R_{x=0}(\vec{v}) = (-v_x, v_y, v_z).
\]
This endomorphism is orthogonal, because \( ||R_{x=0}(\vec{v})|| = ||\vec{v}|| \).

Note that the matrix
\[ A = M(R(x=0),B,B) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]
where \( B = \{i = (1,0,0), j = (0,1,0), k = (0,0,1)\}. \)
**Conclusion**

The matrix of a auto-adjoint endomorphism is a symmetric matrix. Thus, you can always build an auto-adjoint endomorphism.

The orthogonal endomorphisms have great importance in the study of isometries: rotation and reflection, are some examples.

**Assessment**

1. Show that the following matrices represent an orthogonal endomorphism:
   
   \[
   A = \begin{bmatrix}
   1/2 & \sqrt{3}/3 \\
   \sqrt{3}/3 & \sqrt{3}/2 \\
   \end{bmatrix}
   \]
   
   \[
   B = \begin{bmatrix}
   1/2 & \sqrt{3}/2 \\
   \sqrt{3}/3 & 1/2 \\
   \end{bmatrix}
   \]
   
   \[
   C = \begin{bmatrix}
   0.6 & 0.8 \\
   0.8 & -0.6 \\
   \end{bmatrix}
   \]

2. Let \( V \) be a euclidean space of dimension 2. Let \( G = \begin{bmatrix}
   1 & -1 \\
   1 & 3 \\
   \end{bmatrix} \) be the metric matrix with respect to some basis. Let \( f: V \to V \) be the endomorphism defined in the same basis of the matrix \( A = \begin{bmatrix}
   1 & -1 \\
   3 & 2 \\
   \end{bmatrix} \). Define the endomorphism \( f^* \), adjoint of \( f \).

3. In the real vector space \( \mathbb{R}^2 \), let \( v = (-1,2) \).
   
   (a) Determine the reflection of \( v \) with respect to the line \( y = x \).
   
   (b) Determine the rotation of the vector \( v \) by angle \( \theta = \pi/6 \), around the origin.

4. In the real vector space \( \mathbb{R}^3 \), let the vector \( v = (-2,1,3) \).
   
   (a) Determine the reflection of vector \( v \) on the plane \( x = 0 \).
   
   (b) Determine the rotation angle of the vector \( v \) by \( \pi/3 \), around the axis \( z \).

**Summary**

A linear application allows relating different real vector spaces. It can be represented by a linear transformation of image expression through a die, for certain basis of the spaces starting and arrival or images of certain vectors of a particular basis game space.

The core and the application of a linear image calculation allows, among other things, the classification of a linear transformation as the injectivity and abjectivity.

When there is an isomorphism between two vector spaces, one can “transfer” all the properties of a vector space to another.

The orthogonal endomorphisms has much application in isometries study.
Unit Assessment

(1) Let \( f: \mathbb{R}^3 \rightarrow \mathbb{P}^2 \) be a linear transformation by:
\[ f(1,0,1)=x^2-x, \ f(1,1,1)=x^2-1 \] and \( f(0,-1,1)=3x^2-x-2. \)

a. Determine \( f(a,b,c) \), for all \( a,b,c \in \mathbb{R} \).

b. Determine the image of \( f \).

c. Determine \( f^{-1}(-x+1) \).

d. Justify that if \( f \) is or it is not a monomorphism?

Let \( f: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by \( f(x,y,z)=(x+y,0,y-z) \).

(a) Show that it is a linear transformation;

(b) Determine a basis of \( \text{Nuc}(f) \).

(c) Determine a basis of \( \text{Im}(f) \).

(2) Let \( V \) be a real vector space of basis \( B=\{e_1, e_2, e_3\} \), and \( f \) be a endomorphism of \( V \), which with respect to that basis is represented by the matrix \( A=[1 \ 0 \ 1 \ 0 \ 1 \ 2 \ 2 \ 0 \ 2] \).

(a) Determine \( f(e_1-2e_2+4e_3) \).

(b) Show a system of generators of \( \text{Im}(f) \).

(3) In the real vector plane \( \mathbb{R}^3 \), let \( v=(-1,2,1) \).

(a) Determine the rotation of the vector \( v \) by the angle \( \theta=\pi/4 \), around z axis.

(b) Determine the rotation of the vector \( v \), by the angle \( \theta=-\pi/6 \), around the x axis.

Summative Test Drive Linear Transformation

Instructions

The evaluation test has four questions, some points.

Answer each question clearly and justify each step of the solution.

Grading Scheme

Each point or points worth 10 points. It is considered pass for the student who has at least 50% of the total point.
Unit Readings and Other Resources

The readings in this unit are to be found at course level readings and other resources.

- LAY, D.C., Linear Algebra and its Application, Addison-Wesley Production Company, 1994
- https://en.wikipedia.org/wiki/Linear_algebra
Unit 4. Endomorphism diagonalization and matrix: Quadratic Form

Unit Introduction

This unit is intended primarily to introduce the concept of value and associated itself with vector matrix [endomorphism]. The diagonalization of a matrix [ endomorfismo ] as well as the consequences that emerge from it are matters that will occupy practically more than half of the unit. It is intended, a separately analyze the diagonalization of a matrix , and an endomorphism , and ultimately conclude that they are equivalent terms.

In this unit , a vector \( \vec{v} = [(x_1,x_2,...,x_n)\in\mathbb{R}^n, n\in\mathbb{N}, \) is replaced by a new representation - the matrix representation , ie the vector \( \vec{v} \) is now represented by the row vector, \([x_1 \ x_2 \ ... \ x_n\) ], or the column vector, \([[x_1 \ x_2 \ ... \ x_n]]^t\).

Exercises are performed essentially with real square matrices of orders 2 and 3; however , one can propose exercises with arrays of superior orders , but these will be realized with the aid of SCILAB software

Unit Objectives

Upon completion of this unit you should be able to:

- Determine eigenvalues and eigenvectors of a matrix [ endomorphism ]
- Diagonalize a matrix [ endomorphism ].
- Determining the characteristic polynomial of a square matrix and an endomorphism .
- Solving problems algebraic matrix , applying the matrix diagonalization .
- Represent a bilinear form of a matrix
- Determining a quadratic form from a symmetric bilinear form , and vice versa .
Key Terms

Characteristic polynomial of a matrix: Let $A$ be a real square matrix of order $n \in \mathbb{N}$. Let a real square matrix be $M = A - tI_n$, of order $n \in \mathbb{N}$, where $I_n$ is an identity matrix of order $n \in \mathbb{N}$, and $t$ is a real variable. The symmetric matrix $M$ is the matrix $tI_n - A$, and the value of its determinant,

$$
\Delta(t) = \det(tI_n - A) = (-1)^n \det(A - tI_n),
$$

which is a polynomial in terms of $t$ of order $n$, it is called characteristic polynomial of $A$.

Endomorphism of a characteristic Polynomial: Let $f : V \rightarrow V$ be an endomorphism of the real vector space $V$ of dimension $n$, and let $A$ be a matrix of $f$, with respect to an arbitrary basis of $V$. The characteristic polynomial of $f$ is the characteristic polynomial of $A$.

Eigenvalue and eigenvector of a matrix: Let $A$ be a real square matrix of order $n$. A real scalar $\lambda$ is an eigenvalue of $A$ if there exist a nonzero column vector $\vec{v} = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}^t \in \mathbb{R}^n$ such that

$$
A\vec{v} = \lambda \vec{v}.
$$

In this case, the vector $\vec{v}$ is called eigenvector associated with the eigenvalue $\lambda$.

Eigenvalue and eigenvector of an endomorphism: Let $f : V \rightarrow V$ be an endomorphism of a real vector space $V$ of dimension $n$. A real scalar $\lambda$ is said be an Eigenvalue of the endomorphism $f$, if there exist a nonzero vector $\vec{v} \in V$, such that $f(\vec{v}) = \lambda \vec{v}$.

In this case, it said that $\vec{v}$ is an eigenvector of the endomorphism $f$, associated with the eigenvalue $\lambda$.

The set of all value of the eigenvalues of $A$ is called the spectrum of $f$.

Diagonalizable Matrix: Let $A$ be a real square matrix of order $n$. The matrix $A$ is diagonalizable if it is similar to the diagonal matrix.
That is, \( D = P^{-1} AP \), for some regular matrix of order \( n \), \( P \).

**Diagonalizable Endomorphism:** Let \( f: V \to V \) be an endomorphism of a real vector space \( V \) of dimension \( n \). The endomorphism \( f \) is diagonalizable if it is represented by a diagonal matrix of order \( n \),

\[
D = \text{diag}(k_1, k_2, \ldots, k_n) = \begin{bmatrix}
k_1 & 0 & 0 & \cdots & 0 \\
0 & k_2 & 0 & \cdots & 0 \\
0 & 0 & k_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & k_n
\end{bmatrix}.
\]

In other words, \( f \) is diagonalizable if there exist a basis \( B = \{u_1, u_2, \ldots, u_n\} \) of \( V \), such that

\[
\begin{align*}
f(u_1) &= k_1u_1 \\
f(u_2) &= k_2u_2 \\
& \quad \vdots \\
f(u_n) &= k_nu_n
\end{align*}
\]

**Geometric Multiplicity of an Eigenvalue:** It is called geometric multiplicity of \( \lambda \), and is written as \( m_g(\lambda) \), the dimension of eigen subspace \( E_\lambda \).

**Bilinear form:** Let \( V \) be a real vector space. A bilinear form in \( V \) is a transformation \( f: V \times V \to R \) such that for any scalar \( a, b \in R \), for any vector \( v, v_1, v_2, u, u_1, u_2 \) of \( V \) we have:

a. \( f(au_1 + bu_2, v) = af(u_1, v) + bf(u_2, v) \);

b. \( f(u, [av_1 + bv_2]) = af(u, v_1) + bf(u, v_2) \).

**Quadratic Form.** Let \( V \) be a real vector space. A function \( q: V \to R \)

is a quadratic form if \( q(v) = f(v, v) \), for any symmetric bilinear form \( f \).
Learning Activities

Activity 4.1 - Eigenvector associated with eigenvalue, and endomorphism

Introduction

This activity will the concepts of characteristic polynomial introduce eigenvalue and eigenvector of a matrix [ endomorphism ] and some of its properties.

Activity Details

Definition (Characteristic Polynomial of a matrix). Let $A$ be a real square matrix of order $n \in \mathbb{N}$. Let the real square matrix $M = A - tI_n$, of order $n \in \mathbb{N}$, where $I_n$ be an identity matrix of order $n \in \mathbb{N}$, and $t$ is real number. The symmetric matrix $M$ is the matrix $tI_n - A$, and the value of the determinant,

$$\Delta(t) = |tI_n - A| = \begin{vmatrix} -1^n & |A - tI_n| \end{vmatrix},$$

which is a polynomial in terms of $t$ of order $n$, is called characteristic polynomial of $A$.

Theorem (Cayley – Hamilton). All real square matrix is the root of its characteristic polynomial.

Example. Let the real square matrix be $A = \begin{bmatrix} 1 & 3 & 4 & 5 \end{bmatrix}$. The characteristic polynomial of this matrix is:

$$\Delta(t) = |tI_n - A| = \begin{vmatrix} 1-t & 3 & 4 & 5-t \end{vmatrix} = (1-t)(5-t) - 12 = t^2 - 6t - 7.$$ 

by the theorem Cayley - Hamilton, the matrix $A$ is a root of $\Delta(t) = t^2 - 6t - 7$;

Show that:

$$A^2 - 6A - 7I_2 = \begin{bmatrix} 13 & 18 & 24 & 37 \\ -6 & -18 & -24 & -30 \\ -7 & 0 & 0 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0_n.$$ 

Theorem. Similar matrices have the same characteristic polynomial. That is, given a real square matrix $A$, then the matrix $B = P^{-1}AP$ (similar to $A$), for any real regular matrix $P$, has the same characteristic polynomial than $A$.

Note. There is a rule for determining the characteristic polynomial of a square of order 2 and of order 3.

Case 1 (Square Matrix of order 2). Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, be a square real matrix of order 2. Then, the characteristic polynomial of $A$ is:

$$\Delta(t) = t^2 - \text{tr}(A)t + |A|,$$

where $\text{tr}(A)$ is the trace of $A$, and is equal is equal to the sum of principal elements of a square matrix. In this case, $\text{tr}(A) = a_{11} + a_{22}$.

Case 2 (Square Matrix of order 3). Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a real square matrix of order 3. Then, the characteristic polynomial of $A$ is:

$$\Delta(t) = t^3 - \text{tr}(A)t^2 + (A_{11} + A_{22} + A_{33})t - |A|,$$

where, $A_{ii}$, for all $i \in \{1, 2, 3\}$, represent the cofactor of entries (or element) $a_{ii}$ of matriz $A$. 

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Example. Determine the characteristic polynomial of each of the matrix below:

\[ A = \begin{bmatrix} 5 & 3 & 2 & 10 \\ 1 & 2 & 0 & 3 & 2 & 1 & 3 & 9 \end{bmatrix} \]

Characteristic Polynomial of the matrix A:

\[ \Delta(t) = t^2 - \text{tr}(A)t + |A| = t^2 - 15t + 44 \]

Characteristic Polynomial of B:

\[ \Delta(t) = t^3 - \text{tr}(A)t^2 + (A_{11} + A_{22} + A_{33})t - |A| = t^3 - 13t^2 + 31t - 17 \]

Definition (Eigenvalue and Eigenvector of a matrix). Let \( A \) be a real square matrix order \( n \). A real scalar \( \lambda \) is an eigenvalue of \( A \) if there exist a nonzero column vector \( \vec{v} = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}^t \in \mathbb{R}^n \) such that

\[ A \vec{v} = \lambda \vec{v} \]

Theorem (Properties of eigenvalues and eigenvectors). Let \( A \) be a square matrix of order \( n \). Then, the following statements are holds:

1. A real scalar \( \lambda \) is an eigenvalue of \( A \);
2. The matrix \( M = A - \lambda I_n \) is singular;
3. The real scalar \( \lambda \) is the root of the characteristic polynomial \( \Delta(t) \) of \( A \).

Note (Very important!!). The previous theorem states that the eigenvalues of a real square matrix \( A \) are exactly the real roots of the characteristic polynomial \( \Delta(t) \) de \( A \).

A polynomial on the field \( \mathbb{R} \) cannot have roots, so a given matrix then a given real matrix may not have real eigenvalues therefore no eigenvectors.

Example. Determine the eigenvalues of the real square matrix, \( A = \begin{bmatrix} -1/2 & 1/2 & 1/2 & 0 & -1/2 \end{bmatrix} \).

Calculation of the characteristic polynomial. \( A \) is a matrix of order 3, instead of using the definition of the characteristic polynomial to determine it, we will use the following formula:

\[ \Delta(t) = t^3 - \text{tr}(A)t^2 + (A_{11} + A_{22} + A_{33})t - |A| = t^3 + 2t^2 + t \]

Calculation of eigenvalues. To determine the eigenvalues of \( A \), it is sufficient to determine the roots of \( \Delta(t) = t^3 + 2t^2 + t \).

\[ \Delta(t) = 0 \iff t^3 + 2t^2 + t = 0 \iff t = 0 \text{ ou } t = -1 \]

Now, we will determine whether the eigen subspace associated with \( \lambda_1 = 0 \) and \( \lambda_2 = -1 \)

In the case \( \lambda_1 = 0 \). Solve the homogeneous system

Then, \( E_0 = \{(x_3, 0, x_3): x_3 \in \mathbb{R}\} = \{(1, 0, 1)\} \) is the eigen subspace associated with the eigenvalue
In the case $\lambda_2 = -1$. Solve the homogeneous system

$$MX = 0 \iff \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} x_1 = x_3 \\ x_2 = 0 \\ x_3 \in \mathbb{R} \end{cases}$$

Then, $E_{-1} = \{(-x_3, 0, x_3) : x_3 \in \mathbb{R}\} = \langle (-1, 0, 1) \rangle$ is the eigen subspace associated with the eigenvalue $\lambda_2 = -1$.

Note. It is more than evident that the set of eigenvectors of a real square matrix $A$ of order $n$ associated with an eigenvalue $\lambda$, is obtained by eliminating the eigen subspace $E_\lambda$, the zero vector $0 = (0 \ 0 \ \ldots \ 0) \in \mathbb{R}^n$.

Definition (Algebraic and Geometric Multiplicity of an eigenvalue $\lambda$). Let $A$ be a real square matrix of order $n$. If $\lambda$ is an eigenvalue of the matrix $A$, then the algebraic multiplicity of $\lambda$, represented by $m_a(\lambda)$, is the multiplicity of $\lambda$, as a root of the characteristic polynomial of $A$, $\Delta(t)$; while the geometric multiplicity of $\lambda$, represented by $m_g(\lambda)$, is defined the dimension of the eigen subspace $E_\lambda$.

Theorem. Let $A$ be a real square matrix of order $n$, and let $\lambda$ be an eigenvalue of $A$. Then, $m_g(\lambda) \leq m_a(\lambda)$.

Theorem. Suppose that $v_1, v_2, \ldots, v_k$ are eigenvectors of a real square matrix $A$ of order $n$ associated with different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then, $v_1, v_2, \ldots, v_k$ are linearly independent.

Definition (Characteristic Polynomial of an endomorphism). Let $f : V \rightarrow V$ be an endomorphism of a real vector space $V$ of dimension $n$, and let $A$ be a matrix of $f$, with respect to an arbitrary basis of $V$. The characteristic polynomial of $f$ is the characteristic polynomial of $A$.

Note. As two matrices of $f$ are similar, and similar matrices have the same characteristic polynomial, then an endomorphism has a unique characteristic polynomial.

Definition (Eigenvalue and eigenvector of an endomorphism). Let $f : V \rightarrow V$ be an endomorphism of a real vector space $V$ of dimension $n$. A real scalar $\lambda$ is said to be eigenvalue of an endomorphism $f$, if there exist a nonzero vector $v$ such that $f(v) = \lambda v$. In this case, it said $v$ is an eigenvector of an endomorphism $f$ associated with the eigenvalue $\lambda$. The set of all eigenvalue of $A$ is called the spectrum of $f$.

Definition (Eigen Subspace). Let $\lambda$ be an eigenvalue of an endomorphism $f : V \rightarrow V$ of a real vector space $V$ of dimension $n$. Then, the set

$$E_\lambda = \{ v \in V : f(v) = \lambda v \}$$

is a subspace of $V$, is called eigen subspace associated with $\lambda$. 

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Theorem. Let \( f: V \rightarrow V \) be an endomorphism of a real vector space \( V \) of dimension \( n \), and let \( A \) be a matrix of \( f \), with respect to a basis of any \( V \). Then, the following statements hold:

a. \( \lambda \) is an eigenvalue of \( f \);

b. \( \lambda \) is the root of the characteristic polynomial \( \Delta(t) \) de \( f \);

c. The matrix \( M = A - \lambda I_n \) is singular.

Furthermore, as components of the vectors of \( E_\lambda \), with respect to basis considered, are solutions of the homogeneous system \( MX = 0 \), where \( M = A - \lambda I_n \), \( X = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^t \) e \( 0_{(n \times 1)} = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}^t \).

Definition (Geometric and Algebraic of an eigenvalue of an endomorphism). Let \( f: V \rightarrow V \) be an endomorphism of a real vector space \( V \) of dimension \( n \), and let \( \lambda \) be an eigenvalue of \( f \). It is called the geometric multiplicity of \( \lambda \) and is represented by \( m_g(\lambda) \), the dimension of \( E_\lambda \). While, the algebraic multiplicity of \( \lambda \), which is represented by \( m_a(\lambda) \) is the multiplicity of \( \lambda \), as the root of characteristic polynomial of \( f \).

Theorem. Let \( f \) be an endomorphism of a real vector space \( V \) and let \( \lambda \) be an eigenvalue of \( f \). Then, \( m_g(\lambda) \leq m_a(\lambda) \).

Example. Consider the endomorphism \( g \) of \( P^2 \), defined by:

\[
g(x^2+1)=0, \quad g(-x^2+x)=-x-1, \quad g(-x+1)=x^2+x.
\]

a. Determine the eigenvalues of \( g \), and the algebraic multiplicity of each

b. Determine the eigen subspace of \( g \).

c. Determine the geometric multiplicity of each of the eigenvalues obtained.

Paragraph(a). To determine eigenvalues and eigenvectors of the endomorphism \( g \), it is necessary to determine a matrix \( g \), with respect to a particular basis of \( P^2 \). Since \( B = \{x^2+1,-x^2+x,x^2+x \} \) is a basis of \( P^2 \), and these are the given images of each of the vectors of \( B \) of the endomorphism \( g \), then one can determine the matrix of \( g \), with respect to the basis \( B \). With this, it is sufficient to determine the components

\[
g(x^2+1)=0, \quad g(-x^2+x)=-x-1, \quad g(-x+1)=x^2+x,
\]

in the basis \( B \). then:

\[
g(x^2+1)=0 \equiv_B (0,0,0);
\]

\[
g(-x^2+x)=-x-1 \equiv_B (-1,1,0);
\]

\[
g(-x+1)=x^2+x \equiv_B (1,0,-1).
\]

Therefore, the matrix of \( g \) with respect to the basis \( B \) is

\[
A = M(g; B, B) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\]
Since a characteristic polynomial of \( g \) is the characteristic polynomial of \( A \), then we have:

\[
\Delta(t) = \begin{vmatrix}
t & 0 & 0 \\
0 & t+1 & 0 \\
0 & 0 & t+1 \\
\end{vmatrix} = (t + 1)^2.
\]

The eigenvalues of \( g \) are roots of \( \Delta(t) \) that are 0 and -1. \( m_a(0) = 1 \), because it is a simple root of \( \Delta(t) \), and a \( m_a(-1) = 2 \), because it is a double root of \( \Delta(t) \).

The eigenvalues of \( g \) are roots of \( \Delta(t) \) that are 0 and -1. \( m_a(0) = 1 \), because it is a simple root of \( \Delta(t) \), and a \( m_a(-1) = 2 \), because it is a double root of \( \Delta(t) \).

Paragraph (b). To calculate the eigen subspaces, we must solve the system \( MX=0 \), where \( M=A-\lambda I_3 \), for each eigenvalue \( \lambda \). The solutions of each of the systems gives the components of the eigenvectors considered with respect to the basis.

For \( \lambda=0 \), the set of solution of the system \( MX=0 \) is \( S_0=\langle (1,0,0) \rangle \), that is, the eigen subspace \( E_0=\langle x^2+1 \rangle \).

For \( \lambda=-1 \), the set of solution of the system \( MX=0 \) is \( S_{(-1)}=\langle (0,1,0),(0,0,1) \rangle \), that is, the eigen subspace \( E_{(-1)}=\langle -x^2+x,-x+1 \rangle \).

Paragraph (c). \( m_g(0) = 1 \), because \( \text{dim}(E_0)=1 \), and \( m_g(-1) = 2 \), because \( \text{dim}(E_{(-1)})=2 \).

Theorem. Let \( f∶V\rightarrow V \) be an endomorphism of a real vector space \( V \) of dimension \( n \). Suppose that \( v_1, v_2, ..., v_k \) are eigenvectors of \( f \) associated with different eigenvalues \( \lambda_1, \lambda_2, ..., \lambda_k \), also of \( f \). Then, \( v_1, v_2, ..., v_k \) are linearly independent.

Conclusion

The concepts of eigenvalues and eigenvectors of an endomorphism is defined at expense the same concepts in relation to a matrix. This is due to the fact that any endomorphism have a matrix representation. So every property of eigenvalue and eigenvector of a matrix has an equivalent in endomorphism.

Assessment

Let a real square of order 3, \( A=[41-125-2112] \).

Determine eigenvalues of \( A \).

Determine the eigen subspace of \( A \). Specify the maximum number of linearly

Determine the algebraic multiplicity and the geometric multiplicity of each eigenvalue of \( A \).

Let \( f∶P^2\rightarrow P^2 \) be an endomorphism of real vector space \( P^2 \), defined by

\[
f(ax^2+bx+c)=(2a+b-2c)x^2+(2a+3y-4c)x+a+b-c.
\]

Determine the spectrum of \( f \).

Determine the eigen subspaces of \( f \).
Activity 4.2 - Diagonalizable endomorphism (Diagonalization of auto-adjoint endomorphism and symmetric matrix diagonalization) and diagonalizable matrix

**Introduction**

This activity introduces the concepts of diagonalizable matrix and diagonalizable endomorphism as well as their fundamental properties. The case is approached from the diagonalization of a symmetric matrix.

**Activity Details**

Definition (Diagonalizable Matrix). Let A be a real square matrix of order n. The matrix A is diagonalizable if it is similar to a diagonal matrix $D = P^{-1}AP$, for some regular matrix of order n, $P$.

Theorem. Let A be a real square matrix of order n. The matrix A is diagonalizable (or similar to a matrix $D$) if and only if A admits n linearly independent eigenvectors, that is, the exist a basis of $\mathbb{R}^n$, consisting of eigenvectors of A.

In this case, the diagonal entries of $D$, that is $\lambda_1, \lambda_2, ..., \lambda_n$, are the eigenvalues of A, associated with these linearly independent eigenvectors, and $D = P^{-1}AP$, where $P$ is a matrix whose columns are linearly independent eigenvectors.

Example. Let the real square matrix of order 2, $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$, and let the vectors $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. Now, $Av_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, that is, $v_1$ is an eigenvector of A associated with the eigenvalue 1.

$Av_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, that is, $v_2$ is an eigenvector of A associated with the eigenvalue 4.

Since $A$ is a matrix of order 2, and admits 2 linearly independent eigenvectors, then $A$ is similar to the diagonal matrix $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, and $D = P^{-1}AP$, where $P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$, that is, the matrix $A$ is diagonalizable.

Theorem. Let A be a real square matrix of order n. The matrix A is diagonalizable, if and only if the sum of the geometric multiplicity of the eigen subspace associated with the eigenvalues is equal to n.

Example. Let the real square matrix of dimension 3,
The characteristic polynomial of $A$ is:
$$\Delta(t) = t^3 - \text{tr}(A)t^2 + (A_{11} + A_{22} + A_{33})t - \det(A) = t^3 - 6t^2 + 9t - 4.$$  

The eigenvalues of $A$ are $1$, as algebraic multiplicity $2$, and $4$, as algebraic multiplicity $1$, that is:
$$\Delta(t) = (t-1)^2(t-4).$$  

However, the eigen subspaces associated with the eigenvalues are $1$ and $4$ respectively, $E_1 = \langle (1,0,-1), (0,1,0) \rangle$ and $E_2 = \langle (1,1,0) \rangle$. Thus, $m_g(1) = 2$ and $m_g(4) = 1$. As the dimension of $A$ is $3$, and $m_g(1) + m_g(4) = 3$, then $A$ is diagonalizable, and
$$A = P\text{diag}(1,1,4)P^{-1},$$  

where $P = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$, is a regular matrix of order $3$, whose column are the eigenvectors that generate their eigen subspace $E_1$ and $E_2$, because they are linearly independent.

In this case, for all $m \in \mathbb{N}$, $A^m = P\text{diag}(1^m,1^m,4^m)P^{-1} = P\text{diag}(1,1,4^m)P^{-1}$. For example:

$$A^3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 64 & 0 & 63 \\ 63 & 1 & 63 \\ 0 & 0 & 1 \end{bmatrix}. $$

Be
$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. $$

Then, $B^2 = A$.

Note. The diagonalization of a diagonalizable matrix greatly facilitates algebraic with this matrix. If $A$ is a real square diagonalizable matrix, that is, the matrix $A$ is similar to a diagonal matrix.

$$D = \text{diag}(k_1, k_2, \ldots, k_n) = \begin{bmatrix} k_1 & 0 & 0 & \cdots & 0 \\ 0 & k_2 & 0 & \cdots & 0 \\ 0 & 0 & k_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k_n \end{bmatrix}. $$

Then, $D = P^{-1}AP \iff A = PD\text{diag}(k_1, k_2, \ldots, k_n)P^{-1}$, for some regular real matrix $P$ of order $n$. Now:

For all $m \in \mathbb{N}$, $A^m = PD\text{diag}(k_1^m, k_2^m, \ldots, k_n^m)P^{-1}$; 

For all $m \in \mathbb{N}$, for every real polynomial $f(t)$, $f(A) = P\text{diag}(f(k_1), f(k_2), \ldots, f(k_n))$.  

For $k_i \geq 0$, $i \in \{1, 2, 3, \ldots, n\}$, let $B = P\text{diag}(\sqrt{k_1}, \sqrt{k_2}, \ldots, \sqrt{k_n})P^{-1}$. Then $B^2 = A$, and the eigenvalues of $B$ are not negative.
Theorem. Let $A$ be a real square matrix of order $n$. Suppose that the characteristic polynomial of $A$ is $\Delta(t)=(t-a_1)(t-a_2)\cdots(t-a_n)$. Then the real square matrix $A$ is diagonalizable. That is, $A$ is similar to the diagonal matrix.

Definition (Diagonalizable Endomorphism). Let $f:V\rightarrow V$ be an endomorphism of a real vector space $V$ of dimension $n$. The endomorphism $f$ is diagonalizable if it is represented by a diagonal matrix of order $n$.

In other words, $f$ is diagonalizable if there exist a basis $B=\{u_1,u_2,\ldots,u_n\}$ of $V$, such that

Theorem. Let $f:V\rightarrow V$ be an endomorphism of a real vector space $V$ of dimension $n$. $f$ is diagonalizable if and only if there exist a basis of $V$, only composed of eigenvectors of $V$. In this case, $f$ is represented by a diagonal matrix, whose main entries are the eigenvalues associated with these eigenvectors (that constitute a basis of $V$).

Example. The endomorphism $g$ of $P^2$, defined by:
$$g(x^2+1)=0, \enspace g(-x^2+x)=-x-1 \text{ and } g(-x+1)=x^2+x$$
is diagonalizable (see the example discussed earlier under endomorphism).

Theorem. Let $f:V\rightarrow V$ be an endomorphism of an real vector space $V$ of dimension $n$, and let $\Delta(t)$ the characteristic polynomial of $f$. If $\Delta(t)=(t-a_1)(t-a_2)\cdots(t-a_n)$, then the endomorphism $f$ is diagonalizable. That is, $f$ is represented by a diagonal matrix.

Theorem. Let $f:V\rightarrow V$ be an endomorphism of a real vector space $V$ of dimension $n$. The endomorphism $f$ is diagonalizable if the sum of the geometric multiplicity of the eigen subspaces associated with the eigenvalues, is equal to $\dim(V)=n$.

Example. The endomorphism $g$ of $P^2$, defined by:
$$g(x^2+1)=0, \enspace g(-x^2+x)=-x-1 \text{ and } g(-x+1)=x^2+x$$
is diagonalizable (see the example discussed earlier under endomorphism).

Diagonalization of a symmetric matrix.

Theorem. Let $A$ be a real symmetric matrix. Then, all eigenvalue $\lambda$ is real.

Theorem. Let $A$ be a real symmetric matrix of order $n$, and let $u$ and $v$ be eigenvectors of $A$, associated with different eigenvalues, $\lambda_1$ and $\lambda_2$, of $A$. Then, the vectors $u$ and $v$ are orthogonal with respect to the canonical inner product on $\mathbb{R}^n$.

Theorem. Let $A$ be a real symmetric matrix of order $n$. Then $A$ is diagonalizable, that is, there exist an orthogonal matrix $P$, such that $D=P^{-1}AP$ is a diagonal matrix of order $n$.

In this case, it is said that $A$ is orthogonally diagonalizable, and the matrix $P$ obtained by normalizing the orthogonal basis of $\mathbb{R}^n$, composed of eigenvectors of $A$.

Example. Let $A=[2 \ 0 \ -2 \ 5]$ be a real symmetric matrix. Find an orthogonal matrix $P$ such that $P^{-1}AP$ is a diagonal matrix.
The characteristic polynomial of $A$ is $\Delta(t)=t^2-7t+6=(t-1)(t-6)$, that is, the eigenvalues of $A$ are 1 and 6.

$E_1=\langle(2,1)\rangle$ and $E_6=\langle(-1,2)\rangle$. Since $A$ is symmetric, the vectors $v\_1=(2,1)$ and $v\_2=(-1,2)$ are orthogonal with respect to the canonical inner product on $\mathbb{R}^2$.

Normalizing the vectors $v\_1=(2,1)$ and $v\_2=(-1,2)$, obtained the matrix $P=[\sqrt{5}/5 \ 2\sqrt{5}/5 \ -2\sqrt{5}/5 \ \sqrt{5}/5]$.

**Conclusion**

The concepts of diagonalizable matrix and diagonalizable endomorphism are similar. The process that is used to diagonalize a matrix, is also used to diagonalize an endomorphism.

Not every real matrix is diagonalizable, therefore, not every endomorphism is diagonalizable.

However, every symmetric matrix is diagonalizable, more than that, it is orthogonally diagonalizable.

**Assessment**

Let $B=[11 \ -8 \ 4 \ -8 \ -1 \ 4 \ -2 \ -4]$ be a real symmetric matrix of order 3.

Determine the eigenvalues of $B$.

Determine a set with the maximum number of linearly independent eigenvectors.

Determine an orthogonal matrix $P$ such that $D=P^{-1}BP$.

Calculate $B^5$, from the factorization $D=P^{-1}BP$.

Let $f: P^2 \stackrel{\rightarrow}{\longrightarrow} P^2$ be an endomorphism of the real vector space $P^2$, defined by $f(ax^2+bx+c)=(2a+b-2c)x^2+(2a+3y-4c)x+a+b-c$.

Show that $f$ is diagonalizable. If so, determine a basis of $P^2$, composed of only linear independent eigenvectors.

**Activity 4.3 - Bilinear Form**

**Introduction**

This activity introduces the concept of bilinear form, and highlights the concepts of symmetric bilinear forms and quadratic forms.

It introduces the concept of coherent matrix.
**Activity Details**

Definition (bilinear Form). Let \( V \) be a real vector space. A bilinear form in \( V \) is a transformation \( f: V \times V \to \mathbb{R} \) such that for any scalar \( a, b \in \mathbb{R} \), for any vectors \( \mathbf{v} \), \( \mathbf{v}'_1 \), \( \mathbf{v}'_2 \), \( \mathbf{u} \), \( \mathbf{u}'_1 \), \( \mathbf{u}'_2 \) of \( V \) we have:

\[
\begin{align*}
    f(a \mathbf{v} + b \mathbf{v}', \mathbf{v}) &= af(\mathbf{v}, \mathbf{v}) + bf(\mathbf{v}', \mathbf{v}) \\
    f(\mathbf{u}, a \mathbf{v}' + b \mathbf{v}'') &= af(\mathbf{u}, \mathbf{v}') + bf(\mathbf{u}, \mathbf{v}'').
\end{align*}
\]

Example. The canonical inner product in \( \mathbb{R}^n \), \( n \in \mathbb{N} \) and \( n > 1 \) is a bilinear form.

Example. Let \( A = [a_{ij}] \) be a real square matrix of dimension \( n \in \mathbb{N} \) and \( n > 1 \). The transformation \( f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \), defined by \( f(X, Y) = p(x_i, y_i) \), where \( [p(x_i, y_i)] = X^t A Y \), for any column vectors \( X = [x_1, x_2, ..., x_n]^t \), \( Y = [y_1, y_2, ..., y_n]^t \) of \( \mathbb{R}^n \).

So, any inner product in \( \mathbb{R}^n \) is a bilinear form.

Theorem. Let \( f \) be a bilinear form of a real vector space \( V \), and is \( B = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\} \) a basis of \( V \). For any vectors \( \mathbf{u}, \mathbf{v} \in V \), let \( X = [[x_1, x_2, ..., x_n]]^t \) be a column vector consisting of the component of \( \mathbf{u} \) in the basis \( B \), and \( Y = [[y_1, y_2, ..., y_n]]^t \) be a column vector consisting of the components of \( \mathbf{v} \) in the basis \( B \). Then,

\[
[f(\mathbf{u}, \mathbf{v})] = X^t A Y,
\]

where \( A = [f(e_i, e_j)] \), for \( i, j \in \{1, 2, 3, ..., n\} \), representation of the matrix of \( f \), relatively to the basis \( B \), or matrix of \( f \) in \( B \).

Theorem. Let \( V \) be a real vector space of dimension \( n \), let \( f \) be a bilinear form in \( V \), let \( B \) and \( B' \) be two basis of \( V \), and let \( P = M(B \to B') \). then if \( A \) is the matrix of \( f \) in \( B \), then \( A^t = P^t A P \) is a matrix of \( f \) in \( B' \).

Definition (Congruent Matrix). A real square matrix \( B \) is congruent to another, \( A \), and written \( B \cong A \), if there exist a regular matrix \( P \), such that \( B = P^t A P \).

Definition (Rank of bilinear form). The rank of a bilinear form \( f \) of a real vector space \( V \), represented by \( c(f) \), is the rank of a matrix of any matrix that it represent.

Definition (Degenerate and nondegenerate form). Let \( f \) be a bilinear form of the real vector space \( V \). It is said that \( f \) is a degenerate form if \( c(f) < \dim(V) \). It is said that \( f \) is a nondegenerate form if \( c(f) = \dim(V) \).

Symmetric Bilinear Form. Quadratic Form.

Definition (Symmetric Bilinear Form). Let \( f \) be a bilinear form of real vector space \( V \). It is said that \( f \) is a symmetric bilinear form if, for all vectors \( \mathbf{u}, \mathbf{v} \in V \), \( f(\mathbf{u}, \mathbf{v}) = f(\mathbf{v}, \mathbf{u}) \).

Theorem. Let \( f \) be a bilinear form of the real vector space \( V \) of dimension \( n \). Then \( f \) is symmetric if and only if the matrix of \( f \) in basis \( B \) of \( V \), is a symmetric matrix.

Theorem. Let \( f \) be a symmetric bilinear form of a real vector space \( V \) of dimension \( n \). There exist a basis \( B = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\} \) of \( V \) such that \( f \) is represented by a diagonal matrix.
Theorem. Let $A$ be a real symmetric matrix. Then $A$ is congruent with a diagonal matrix, that is, there exist a regular matrix $P$ such that $P^t AP$ is a diagonal matrix.

Note. There exist an algorithm for determining a diagonal congruent matrix to a finite symmetric matrix. The interested reader can consult the book "Linear Algebra", De Seymour Lipschutz and Marc Lipson (3ª ed.), p. 379.

Definition (Quadratic Form). Let $V$ be a real vector space. A function $q: V \rightarrow \mathbb{R}$ is a quadratic form if $q(\vec{v}) = f(\vec{v}, \vec{v})$, for any symmetric bilinear form $f$.

Note. One can determine a bilinear form $f$ from a quadratic form $q$, from the following form called polar form of $f$:

$$f(u, v) = \frac{1}{2}[q(u + v) - q(u) - q(v)].$$

Suppose that $f$ is represented by a symmetric matrix $A = [a_{ij}]$. Let

$$X = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^t$$

be a column vector represented by variable. Then $q$ can be represented by:

$$q(X) = f(X) = X^t AX = \sum_{(i,j)} a_{ij} x_i x_j = \sum_i a_{ii} x_i^2 + 2\sum_{i<j} a_{ij} x_i x_j.$$ 

Thus, a quadratic form $q$ with variables $x_1, x_2, \ldots, x_n$ is a polynomial in which all terms are of degree 2.

Definition (Symmetric positive bilinear form and positive semi-definite). A symmetric bilinear form of a real vector space $V$ is said:

- Positive definite if $q(\vec{v}) = f(\vec{v}, \vec{v}) > 0$, for all vector $\vec{v} \neq 0$ of $V$;
- Non negative semi-definite if $q(\vec{v}) = f(\vec{v}, \vec{v}) \geq 0$, for all vector $\vec{v}$ of $V$.

Conclusion

There exist various bilinear form; one way or the other, for the properties.

All symmetric bilinear form is represented by a symmetric matrix, and can be represented by a diagonal matrix, consequently, any symmetric matrix is congruent to a diagonal matrix.

A quadratic form $q$ on variables $x_1, x_2, \ldots, x_n$ is a polynomial in which all terms are of degree two.

One can obtain a quadratic form from a symmetric bilinear form, and vice versa.

Assessment

Let the bilinear form of the real vector space $\mathbb{R}^3$, defined by

$$f(u, v) = 3x_1 y_1 - 2x_1 y_3 + 5x_2 y_1 + 7x_2 y_2 - 8x_2 y_3 + 4x_3 y_2 - 6x_3 y_3,$$

for any vectors $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ of $\mathbb{R}^3$. 

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Determine the matrix of $f$ with respect to canonical basis of $\mathbb{R}^3$.

Let the bilinear form of the real vector space $\mathbb{R}^2$, defined by

$$f((x_1,x_2),(y_1,y_2))=2x_1 y_1-3x_1 y_2+4x_2 y_2.$$ 

Determine the matrix $A$ of $f$ with respect to basis $\{u_1=(1,0),u_2=(1,1)\}$.

Determine the matrix $B$ of $f$ with respect to basis $\{v_1=(2,1),v_2=(1,-1)\}$.

Determine a regular matrix $P$ of order 2, such that $B=P^t AP$.

Determine the symmetric matrix that represented the following quadratic form of $\mathbb{R}^3$:

$$q(x,y,z)=3x^2+4xy-y^2+8xz-6yz+z^2.$$ 

Obtain the symmetric bilinear form $f$ from $q$.

### Summary

The diagonalization of a matrix [ endomorphism ] depends largely on the number of linearly independent eigenvectors , compared to the size of $A$ [ dimension of the space endomorphism ] . A quadratic form is obtained from a symmetric bilinear form .

### Unit Assessment

Check your understanding!

Summary test Diagonalization unit and endomorphism of a matrix, quadratic form

Instructions

The evaluation test has four questions and some points.

Answer each question clearly and justifying each step of solution.

### Grading Scheme

Each questions worth 10 points. It is considered past the student who has at least 50 % of the total.
Feedback

- Let $A=[3 -1 1 7 -5 1 6 -6 2]$ be a square real matrix of order 3.
- Determine the eigenvalues of $A$.
- Determine eigen subspace of $A$. Conclude if $A$ is or is not diagonalizable.
- Consider the endomorphism of the real vector space $P^2$, defined by:
- Show that $f$ is diagonalizable.
- Determine a basis of $f$ such that $f$ is represented by a diagonal matrix.
- Determine a real matrix $A$ of $f$, different from diagonal matrix.
- Determine a real matrix $B$ of $f$ such that $B^2=A$. Determine $B^5$.
- Let $A=[1 -1 1 -1 2 2 1 2 -1]$ be a symmetric matrix of order 3.
- Diagonalize orthogonally the matrix $A$.
- Determine the symmetric matrix that represent the following quadratic form of $R^3$ $q(x,y,z)=3x^2+xz-2yz$.
- Determine the symmetric bilinear form $f$ from $q$.

Unit Readings and Other Resources

The readings in this unit are to be found at course level readings and other resources.

- LAY, D.C., Linear Algebra and its Application, Addison-Wesley Production Company, 1994

Optional readings and other resources:

https://en.wikipedia.org/wiki/Linear_algebra

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